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TIME-FREQUENCY AND PULSE PROPAGATION IN DISPERSIVE MEDIA

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13. ABSTRACT *(Maximum 200 Words)*
We apply the methods of time-frequency analysis to pulse propagation in a dispersive medium. We give exact expressions for conditional moments and in particular for the contraction and spreading of the pulse, the covariance between position and wave number and other relevant physical quantities. For the contraction time it is shown that the important quantity is the initial correlation between position and group velocity. A simple physical and mathematical model is presented that explains why sometimes pulses contract before eventually expanding. In addition, we give formulas for the calculation of the instantaneous frequency of a pulse at a given position which show how dispersion effects instantaneous frequency. Also, we apply the Wigner distribution to study pulse propagation in dispersive media. Doing so provides insight and simplifies the view of pulse propagation and further leads to simple approximation methods that evolve a pulse in time. In addition, we derive representations that jointly involve the four variables of time, frequency, space, and spatial frequency, and we apply these representations to wave propagation. A number of examples are used to illustrate the formulas derived.

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1 Introduction

Waves and particles are the main constituents of the world but waves are our main method of communication and detection, whether biological or technological. Historically, the study of waves has been separated into the stationary (standing wave) and nonstationary (pulse) case. Pulses are of fundamental consideration in radar, sonar, acoustics, fiber optics, among many other areas. A pulse has been called by many terms: transient, wave group, progressive wave, wave packet, nonrecurrent wave, traveling wave, non-periodic wave have all been used to describe basically the same thing. Because one of the main properties of pulse propagation is *dispersion*, that is, the fact that different frequencies propagate with different speeds, time-frequency analysis offers an ideal approach toward their study [3, 4]. Over the past two decades or so there has been substantial progress in the development of new methods for analyzing a signal jointly in time and frequency and this has led to new results in the basic nature of pulses.

In this report we deal with pulses and one of the main aims is to give a simple view of pulse propagation, an approach that we call "a local view"which makes pulses easier to understand and deal with intuitively. We have devised a new method to study pulse propagation in dispersive media and using this approach we have derived exact expressions for many physical quantities such as the spreading of a pulse, the conditions as to when a pulse contracts, the contraction time, and other important physical quantities. We also consider higher order dispersion, which is when there are terms on the dispersion relation that are higher than quadratic, a concept that will be explained subsequently.

We list the main ideas, results, and methods presented in this report:

1. We describe a method to calculate exact moments of a propagating pulse without having to calculate the pulse itself.

2. We discuss a local view of pulse propagation that explains in simple terms many of the curious results of pulse propagation.

3. We have introduced the concept of covariance for pulses which gives a clear picture of the physical situation and clarifies many of the mathematical results obtained.

4. We have studied higher order dispersion using the results of the method of point 1.

5. We show how instantaneous frequency changes from place to place for a propagating pulse.

6. We have explicitly given results for two standard situations, when a pulse is known for all space at a given time and when a pulse is generated at a particular place for all time. Both situations can be understood using the local view model.

7. We give many exactly solvable examples.

8. We show that the Wigner distribution approach to pulse propagation is very fruitful both from a physical and numerical point of view.

Organization of this report: We have attempted to make this report basic in regards to the issues presented and hence we discuss some of the fundamental facts regarding pulse propagation, wave equations, and dispersion. In this manner the report may be read by someone who is not particularly familiar with pulse propagation and dispersion.

2 Wave Equations and Dispersion

The concept of dispersion comes about in the following way[12, 17]. Take the standard free space wave equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

and examine how the simplest wave propagates

$$u_S(x,t) = e^{ikx - i\omega t} \tag{2}$$

If one takes an instant snapshot of Eq. (2), that is if one looks at it for a fixed time, then what one has is a spatial wave and its spatial frequency is given by k. On the other hand if we stay at one position then what we have is a wave in time and ω is the frequency. Therefore both k and ω are frequencies, one a spatial frequency and the other a time frequency. In Eq. (2) one can chose k and ω independently. However, what the wave equation does is force a relation between the two and that is called the dispersion relation. It is also important to emphasize that by putting Eq. (2) into a wave equation we are not trying to find a solution but just seeing if such a solution is possible at all. Also, we point out that the importance of considering Eq. (2) is not only that it is physically simple, but because any pulse can be decomposed in terms of it, where k and ω range over all space. Hence knowing what it does for k and ω allows one to find the general solution. This was one of the great achievements in pulse propagation and is discussed in subsequent sections.

We also should point out that getting the wave equations, Eq. (1), was one of the major problems of the 18th century. It was obtained by D'Alembert and Euler and each had a different view point as to what is an acceptable solution. It started a major debate that lasted 50 years and led to many new ideas including Fourier analysis.

If one substitutes Eq. (2) into Eq. (1) one gets that it can only be a solution if there is a relation between k and ω. In particular

$$\omega^2 = c^2 k^2 \tag{3}$$

or

$$\omega = \pm ck \tag{4}$$

Now we see that one cannot chose k and ω at will, we must have that relation satisfied. Let us momentarily choose $\omega = k$ and therefore

$$u_S(x,t) = e^{ik(x-ct)} \tag{5}$$

and we see that for this case the wave propagates to the right. Moreover one can now show that if we have any pulse at $t = 0$ given by $u(x,0)$, then

$$u(x,t) = u(x-ct,0) \tag{6}$$

That is, it propagates without distortion. That Eq. (6) follows is because an arbitrary wave can be decomposed into simple waves and each travels at the same speed.

We also point out that one can have a different perspective and write

$$u_S(x,t) = e^{i\omega(x/c-t)} \tag{7}$$

and we will discuss this later.

Now consider the beam equation

$$\frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0 \tag{8}$$

Again putting in the simple solution one obtains that

$$\omega^2 = \alpha^2 k^4 \tag{9}$$

and again let use choose $\omega = \alpha k^2$. Now

$$\begin{aligned} u_S(x,t) &= e^{ik-i\alpha k^2 t} && (10) \\ &= e^{ik(x-\alpha k t)} && (11) \end{aligned}$$

One still has the possibility of a simple wave but now the velocity is αk, that is the velocity depends on k and therefore if we had the sum of two waves they would not move in unison.

Hence if we have a pulse that is the sum of simple pulses there is no one velocity associated with the whole pulse and moreover it is not the case that one can find a velocity so that Eq. (6) is true. There are other velocities that can be associated with the pulse and that is something we will discuss in detail later.

Generally speaking suppose we have an arbitrary wave equation, put the simple solution in and obtain a relation

$$\omega = W(k) \tag{12}$$

then

$$u_S(x,t) = e^{ik\left(x - \frac{W(k)}{k}t\right)} \tag{13}$$

and we see that a simple wave with wave number k propagates with a velocity

$$v_p = \frac{W(k)}{k} \tag{14}$$

and this is called the phase velocity. If we only consider simple waves this would be it. However if we consider combination of pure waves, that is a pulse, these considerations are not sufficient. In pulse propagation there is an equal and perhaps more important velocity, the group velocity, as we will explain shortly.

2.1 General Solution

Linear partial differential equations whose solutions give wave like behavior come in many varieties, the above two being prime examples. Fortunately, the solution to all of them can be written in a simple form. All linear wave equations with constant coefficients may be written in this form

$$\sum_{n=0}^{N} a_n \frac{\partial^n u}{\partial t^n} = \sum_{n=0}^{M} b_n \frac{\partial^n u}{\partial x^n} \tag{15}$$

One attempts to solve it by substituting $e^{ikx - i\omega t}$ into Eq. (15) to give

$$\sum_{n=0}^{N} a_n (-i\omega)^n = \sum_{n=0}^{M} b_n (ik)^n \tag{16}$$

or

$$\sum_{n=0}^{N} a_n (-i\omega)^n - \sum_{n=0}^{M} b_n (ik)^n = 0 \tag{17}$$

One can solve for ω in terms of k

$$\omega = W(k) \tag{18}$$

or solve for k in terms of ω

$$k = K(\omega) \tag{19}$$

Which one to choose depends on the formulation of the problem and specifically on the initial conditions. The two basic initial physical situations, that we call case A and case B are

$$\text{Initially Given} : u(x,0) \qquad \text{(Case A)} \tag{20}$$
$$: u(0,t) \qquad \text{(Case B)} \tag{21}$$

The first case is when we have the spatial wave at a given time, and the second when we have the wave at a given position for all time. An example of the first is if we pluck a string and let go at time zero. An example of the second is if we are at a fixed position and create a pulse, for example, a radar, sonar, or fiber optic pulse.

We first discuss Case A. The general solution is [12, 17]

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,0)\, e^{ikx - iW(k)t}\, dk \tag{22}$$

where $S(k,0)$ is the initial spatial spectrum and is calculated from the initial pulse

$$S(k,0) = \frac{1}{\sqrt{2\pi}} \int u(x,0)\, e^{-ikx}\, dx \tag{23}$$

and $W(k)$ is the dispersion relation as discussed above. The general solution given by Eq. (22) can be found in textbooks [12, 17].

If one defines the time dependent spectrum by

$$S(k,t) = S(k,0)\, e^{-iW(k)t} \tag{24}$$

then $u(x,t)$ and $S(k,t)$ form Fourier transform pairs between x and k

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,t)\, e^{ikx}\, dk \tag{25}$$
$$S(k,t) = \frac{1}{\sqrt{2\pi}} \int u(x,t)\, e^{-ikx}\, dx \tag{26}$$

It is crucial to note that $u(x,t)$ and $S(k,t)$ form Fourier transform pairs between k and x for *all time.* Case B will be developed in the chapter called Case B.

Important Concepts

There are a number of important concepts that arise and we discuss them here briefly.
Modes. As we have seen from the examples above there can be many solutions to the dispersion relation and each solution is called a mode. For example for the beam equation we have that

$$W(k) = \alpha k^2 \qquad W(k) = -\alpha k^2 \tag{27}$$

and each solution is called a mode. The general solution is then the sum of the modes. In this report we will be discussing one mode at a time.

Reality and attenuation. If for a particular mode $W(k)$ is real, then there is no attenuation. The reason for that is that suppose $W(k)$ was complex

$$W(k) = W_R(k) - iW_I(k) \tag{28}$$

then we would have

$$e^{ikx-iW(k)t} = e^{ikx-i(W_R(k)-iW_I(k))t} \tag{29}$$

$$= e^{-W_I(k)t}e^{ikx-iW_R(k)t} \tag{30}$$

and we see that because of the term $e^{-W_I(k)t}$ the wave would dye out. Therefore, depending on whether $W(k)$ is complex or real we will have damping or not. In this report we consider the case where we have no damping. However we emphasize that the damped case is very important. See the "Future Work" section of this report.

Group velocity. A central idea in the study of pulse propagation is the group velocity, $v(k)$, which is defined by

$$v(k) = W'(k) \tag{31}$$

There are many plausible arguments that have been given in the literature for calling this quantity the group velocity and most books give a plausible argument for defining the above as a "velocity". In a subsequent section we will give new relations for a propagating pulse that we think gives a very clear picture as to the physical meaning of $v(k)$ is and how it is related to the propagation of the center of mass of the pulse. The word group is somewhat of a misnomer but it comes about from the original derivation as historically given and indicates how a narrowband set of waves (a group of waves) centered about the spatial frequency given by k propagates.

Transit time. In the chapter Case B we will study the time properties of a pulse at a fixed position. We will see that the natural quantity that appears there is

$$\tau(\omega) = K'(\omega) \tag{32}$$

We will see that it is related to the amount of time it takes for a frequency to travel a unit distance. We call it the frequency transit time.

Structural/geometrical and Media dispersion. As we have seen above for the free space wave equation there is no dispersion. But that there is dispersion for the beam equation. However, if one imposes boundary conditions on the free space wave equations, such as in a waveguide, then dispersion will occur because the imposition of the boundary conditions generally forces a relationship between ω and k. This type of dispersion is called geometric or structural dispersion in contrast to media dispersion which comes from the wave equation, such as the beam equation. The reason one has dispersion in such a case is because there is reinforcement, and cancellations of waves depend on the geometry and therefore certain modes die out and others survive.

2.2 Asymptotic Solution

One of the fundamental tools for linear wave propagation has been the so called asymptotic solution, first obtained by Kelvin using the method of stationary phase. We give it here because most of our work will be to obtain exact equations, but it will be interesting to contrast with the traditional asymptotic solution. We do not give the derivation here as it can be found in most books. One takes the dispersion relation for each mode and sets

$$W'(k) = \frac{x}{t} \tag{33}$$

and solves it for k which we call k_s

$$k_s = k_s(x/t) \tag{34}$$

These are the so-called stationary points. The asymptotic solution is then [12, 17]

$$u_a(x,t) \sim S(k_s,0)\sqrt{\frac{1}{tW''(k_s)}}e^{ik_s x - iW(k_s)t - i\pi \operatorname{sgn} W''/4} \tag{35}$$

$$\begin{aligned} u_a(x,t) &\sim \frac{S(k_s,0)}{\sqrt{tW''(k_s)}}e^{ik_s x - iW(k_s)t - i\pi \operatorname{sgn} W''/4} && (36)\\ &= A_a(x,t)e^{i\varphi_a(x,t)} && (37)\end{aligned}$$

For later convenience we define the amplitude and phase

$$\begin{aligned} A_a(x,t) &= \frac{S(k_s,0)}{\sqrt{tW''(k_s)}} && (38)\\ \varphi_a(x,t) &= k_s x - W(k_s)t - i\pi \operatorname{sgn} W''/4 && (39)\end{aligned}$$

A simple derivation and application of this to filtered signals was given by Cohen [9].

2.3 Pulse Propagation As a Filtering Operation

The characteristic of pulse propagation is that the spatial spectrum has the form

$$S(k,t) = S(k,0)\ e^{-iW(k)t} \tag{40}$$

It is a product of the initial spatial spectrum multiplied by $e^{-iW(k)t}$. But this is standard filtering and of course in the time domain it is the convolution of the initial pulse with the function that produces the spatial spectrum $e^{-iW(k)t}$. If we define

$$w(x,t) = \frac{1}{\sqrt{2\pi}} \int e^{-iW(k)t}\, e^{ikx}\, dk \tag{41}$$

then $u(x,t)$ is the convolution of $u(x,0)$ with $w(x,t)$. In particular,

$$u(x,t) = \int u(x-x',0) w(x',t) dx' \tag{42}$$

$$= \int u(x',0) w(x-x',t) dx' \tag{43}$$

and of course this is the classical expression of the Green's function approach. The reason we mention it here is that one can write a fast and simple program to calculate $u(x,t)$ because convolution can be done by the FFT algorithm. Dr. Patrick Loughlin [13] has written a simple program using this ideas that we have found very useful to study the issues discussed in the report.

3 Exact Moments of a Pulse

We define the moments of the pulse in the standard way

$$\langle\, x^n \,\rangle_t = \int x^n\, |u(x,t)|^2\, dx \tag{44}$$

and our aim is to show that one can calculate the exact moments even though we do not solve for $u(x,t)$. There are three reasons for this:

- First, it is always the case that one can calculate these moments from the spectrum, that is

$$\langle\, x^n \,\rangle_t = \int x^n\, |u(x,t)|^2\, dx \tag{45}$$

$$= \int S^*(k,t)\, \mathcal{X}^n\, S(k,t)\, dk \tag{46}$$

  where $\mathcal{X}$ is the position operator in the k representation

$$\mathcal{X} = i\, \frac{\partial}{\partial k} \tag{47}$$

- Secondly, and what potentially offers a simple method of calculating moments, is the fact that

$$S(k,t) = S(k,0)\, e^{-iW(k)t} \tag{48}$$

  and therefore the differentiation indicated by $\mathcal{X}^n\, S(k,t)$ can actually be carried on.

- Third, once the differentiation is done, the resulting quantities are expressible in terms of polynomials in time and $S(k,0)$ which in turn means we can express things in terms of the initial quantities.

We now give the results for some important low order moments and discuss the physical meaning. The derivations are given in the appendix.

3.1 The Mean

The first conditional moment is

$$\langle\, x \,\rangle_t = \int x\, |u(x,t)|^2 \, dx \tag{49}$$

$$= \int S^*(k,t)\, \mathcal{X}\, S(k,t)\, dk \tag{50}$$

and evaluates to

$$\langle\, x \,\rangle_t = \langle\, x \,\rangle_0 + \; Vt \tag{51}$$

where

$$V = \int v(k)\, |S(k,0)|^2 \, dk \tag{52}$$

$$v(k) = \frac{dW(k)}{dk} \tag{53}$$

We now discuss the physical meaning of Eq. (51):

a) First, we point and emphasize that the usual definition of group velocity, Eq. (53), appears in a natural way as can be seen from the derivation in the appendix. It does not have to be imposed in any way.

b) The center of mass travels with a constant velocity, V, for all wave equations and all situations, and all starting conditions.

c) The velocity, V, depends of course on the dispersion relation. However it also depends on the initial spectrum. Therefore how one starts the pulse effects the propagation of the center of mass!

d) Also, V depends only on the magnitude of the initial spectrum.

e) Note that V can be considered an average with the weighing function $|S(k,0)|^2$. It is the average of the group velocity. Therefore $v(k)$ can be thought of as the velocity for each value of k and we will sometimes write

$$\langle\, v \,\rangle = \int v(k)\, |S(k,0)|^2 \, dk = V \tag{54}$$

3.2 Second Moment

$$\langle x^2 \rangle_t = \int x^{\,2} |u(x,t)|^2 \, dx \tag{55}$$

$$= \int S^*(k,t)\, \mathcal{X}^2 \, S(k,t)\, dk \tag{56}$$

and evaluates to

$$\langle x^2 \rangle_t = \langle x^2 \rangle_0 + t \, \langle v\mathcal{X} + \mathcal{X}v \rangle_0 + t^2 \, \langle v^2 \rangle \tag{57}$$

Where

$$\langle v\mathcal{X} + \mathcal{X}v \rangle_t = \int S^*(k,t) \; [v(k)\mathcal{X} + \mathcal{X}v(k)] \; S(k,t)\, dk \tag{58}$$

$$= \int S^*(k,t) \; \left[v(k) i \frac{\partial}{\partial k} + i \frac{\partial}{\partial k} v(k) \right] \; S(k,t)\, dk \tag{59}$$

$$= i \int S^*(k,t) \; \left[2v(k) \frac{\partial S(k,t)}{\partial k} + v'(k) S(k,t) \right] \; dk \tag{60}$$

A standard notation for such quantities is the anticommutator

$$[v, x]_+ = v(k)\mathcal{X} + \mathcal{X}v(k) \tag{61}$$

Also this quantity can be calculated from the pulse directly. If we take

$$\mathcal{K} = -i \frac{\partial}{\partial x} \tag{62}$$

then also

$$\langle v\mathcal{X} + \mathcal{X}v \rangle_t = \int u^*(x,t) \; [v(\mathcal{K})x + xv(\mathcal{K})] \; u(x,t)\, dx \tag{63}$$

This quantity and its physical interpretation will be quite important and we emphasize that it comes up naturally in the derivation. Alternative expressions will be given in the next section, where we will also discuss its physical meaning. Also, in Eq. (57), $\langle v^2 \rangle$ is obtained from

$$\langle v^2 \rangle = \int v^2(k) \, |S(k,0)|^2 \, dk \tag{64}$$

3.3 Spread

The spread at a particular time, defined in the usual way by

$$\sigma^2_{x|t} = \langle x^2 \rangle_t - \langle x \rangle_t^2 \tag{65}$$

$$= \int S^*(k,t)\, (\mathcal{X} - \langle x \rangle_t)^2 \, S(k,t)\, dk \tag{66}$$

works out to be remarkably simple,

$$\sigma_{x|t}^2 = \sigma_{x|0}^2 + 2\,t\,\mathrm{Cov}_{xv|0} + t^2\sigma_v^2 \tag{67}$$

where

$$\sigma_v^2 = \langle\, v^2 \,\rangle - \langle\, v \,\rangle^2 \tag{68}$$

$$= \int (v(k) - V)^2\, |S(k,0)|^2\, dk \tag{69}$$

$$\mathrm{Cov}_{xv|0} = \tfrac{1}{2}\langle\, v\mathcal{X} + \mathcal{X}v \,\rangle_0 - \langle\, v \,\rangle_0 \langle\, x \,\rangle_0 \tag{70}$$

We now make some remarks regarding the above expression for the spread:

1. Notice that σ_v^2 is the standard deviation of the group velocity. Why should that come in? Notice that it is calculated at the initial time.

2. As is the quantity $\langle\, v\mathcal{X} + \mathcal{X}v \,\rangle_0$ basic to our considerations so is the quantity $\mathrm{Cov}_{xv|0}$; we call it the "covariance" between position and group velocity because it acts very much like the standard covariance. It is also calculated at the initial time.

3.4 Covariance

The covariance plays a basic role in pulse propagation and we now discuss it in more detail. In the above we defined it at time zero but now we define these quantities for an arbitrary time. In particular we define

$$\langle\, xv \,\rangle_t = \tfrac{1}{2}\langle\, v\mathcal{X} + \mathcal{X}v \,\rangle_t \tag{71}$$

$$= \tfrac{1}{2}\int S^*(k,t)\,(v\mathcal{X} + \mathcal{X}v)\,S(k,t)\,dk \tag{72}$$

If the spectrum is written in terms of amplitude and phase

$$S(k,t) \;= |S(k,t)|\, e^{i\psi(k,t)} \tag{73}$$

then, substituting Eq.(73) into Eq. (71) one obtains that

$$\langle\, xv \,\rangle_t = -\int v(k)\,\frac{\partial\psi(k,t)}{\partial k} |S(k,t)|^2\, dk \tag{74}$$

Consider how $\langle\, xv\rangle_t$ changes in time for a pulse. The time dependent spatial spectrum is given by

$$S(k,t) = |S(k,t)|\, e^{i\psi(k,t)} \tag{75}$$

$$= S(k,0)\, e^{-iW(k)t} \tag{76}$$

$$= |S(k,0)|\, e^{i\psi(k,0)} e^{-iW(k)t} \tag{77}$$

$$= |S(k,0)|\, e^{i\psi(k,0) - iW(k)t} \tag{78}$$

and hence the phase evolves as

$$\psi(k,t) = \psi(k,0) - W(k)t \tag{79}$$

Differentiating,

$$\frac{\partial\psi(k,t)}{\partial k} = \frac{\partial\psi(k,0)}{\partial k} - W'(k)t \tag{80}$$

or

$$\frac{\partial\psi(k,t)}{\partial k} = \frac{\partial\psi(k,0)}{\partial k} - v(k)t \tag{81}$$

Substituting this into Eq. (74) we have

$$\begin{aligned}
\langle\, xv\,\rangle_t &= -\int v(k)\,\frac{\partial\psi(k,t)}{\partial k}|S(k,t)|^2\,dk && (82)\\
&= -\int v(k)\left[\frac{\partial\psi(k,0)}{\partial k} - v(k)t\right]\,|S(k,t)|^2\,dk && (83)\\
&= -\int v(k)\,\frac{\partial\psi(k,0)}{\partial k}\,|S(k,t)|^2\,dk + \int v(k)v(k)t\,|S(k,t)|^2\,dk && (84)
\end{aligned}$$

and hence

$$\langle x\, v\,\rangle_t = \langle x\, v\,\rangle_0 + \langle\, v^2\,\rangle t \tag{85}$$

Now consider the covariance

$$\mathrm{Cov}_{xv|t} = \langle\, xv\rangle_t - \langle\, x\rangle_t\langle\, v\rangle_t \tag{86}$$

First note the following

$$\begin{aligned}
\langle\, v\rangle_t &= \int v(k)|S(k,t)|^2\,dk && (87)\\
&= \int v(k)|S(k,0)|^2\,dk && (88)\\
&= \langle\, v\rangle_0 && (89)
\end{aligned}$$

Therefore, the average group velocity does not change and we have,

$$\begin{aligned}
\mathrm{Cov}_{xv|t} &= \langle\, xv\rangle_t - \langle\, x\rangle_t\langle\, v\rangle_t && (90)\\
&= \langle x\, v\,\rangle_0 + \langle\, v^2\,\rangle t - (\langle\, x\,\rangle_0 +\ Vt)\langle\, v\rangle_0 && (91)\\
&= \langle x\, v\,\rangle_0 - \langle\, x\,\rangle_0\langle\, v\rangle_0 + t(\langle\, v^2\,\rangle - V\langle\, v\rangle_0) && (92)\\
&= \langle x\, v\,\rangle_0 - \langle\, x\,\rangle_0\langle\, v\rangle_0 + t(\langle\, v^2\,\rangle - \langle\, v\rangle_0^2) && (93)
\end{aligned}$$

Hence

$$\mathrm{Cov}_{xv|t} = \mathrm{Cov}_{xv|0} +\ \sigma_v^2 t \tag{94}$$

Since the coefficient of t is manifestly positive we see that no matter how negative the initial covariance is, the covariance must eventually turn positive. Why should that be so? What are the implications of that? We will see that with the model we subsequently present these questions will become easy to answer.

Now consider the correlation coefficient between x and v,

$$\rho_{xv|t} = \frac{\mathrm{Cov}_{xv|t}}{\sigma_{x|t}\sigma_{v|t}} = \frac{\mathrm{Cov}_{xv|0} + t\sigma_v^2}{\sigma_v\sqrt{\sigma_{x|0}^2 + 2\,t\,\mathrm{Cov}_{xv|0} + t^2\sigma_v^2}} \tag{95}$$

We now examine how it behaves for large and small times. For large times we have

$$\rho_{xv|t} = \frac{(\mathrm{Cov}_{xv|0} + t\sigma_v^2)}{t\sigma_v^2}\left[1 + \frac{\sigma_{x|0}^2 + 2\,t\,\mathrm{Cov}_{xv|0}}{t^2\sigma_v^2}\right]^{-1/2} \tag{96}$$

$$= \left[1 + \frac{\mathrm{Cov}_{xv|0}}{t\sigma_v^2}\right]\left[1 + \frac{\sigma_{x|0}^2 + 2\,t\,\mathrm{Cov}_{xv|0}}{t^2\sigma_v^2}\right]^{-1/2} \tag{97}$$

$$\backsim \left[1 + \frac{\mathrm{Cov}_{xv|0}}{t\sigma_v^2}\right]\left[1 - \frac{1}{2}\frac{\sigma_{x|0}^2 + 2\,t\,\mathrm{Cov}_{xv|0}}{t^2\sigma_v^2} + \frac{3}{8}\left(\frac{\sigma_{x|0}^2 + 2\,t\,\mathrm{Cov}_{xv|0}}{t^2\sigma_v^2}\right)^2 \cdots\right] \tag{98}$$

$$= \left[1 + \frac{\mathrm{Cov}_{xv|0}}{t\sigma_v^2}\right]\left[1 - \frac{\mathrm{Cov}_{xv|0}}{t\sigma_v^2} - \frac{1}{2}\frac{\sigma_{x|0}^2}{t^2\sigma_v^2} + \frac{3}{8}\left(\frac{4\,\mathrm{Cov}_{xv|0}^2}{t^2\sigma_v^4}\right) \cdots\right] \tag{99}$$

$$= \left[1 + \frac{\mathrm{Cov}_{xv|0}}{t\sigma_v^2}\right]\left[1 - \frac{\mathrm{Cov}_{xv|0}}{t\sigma_v^2} + \frac{3\,\mathrm{Cov}_{xv|0}^2 - \sigma_{x|0}^2\sigma_v^2}{2t^2\sigma_v^4} \cdots\right] \tag{100}$$

$$= \left[1 + \frac{2\,\mathrm{Cov}_{xv|0}^2 - \sigma_{x|0}^2\sigma_v^2}{2t^2\sigma_v^4} \cdots\right] \tag{101}$$

As time goes to infinity we have that

$$\rho_{xv|t} \to 1 \qquad \text{as} \qquad t \to \infty \tag{102}$$

which shows that at infinity there is perfect correlation between group velocity and position!

For small times

$$\rho_{xv|t} = \frac{\text{Cov}_{xv|0} + t\sigma_v^2}{\sigma_v \sqrt{\sigma_{x|0}^2 + 2\,t\,\text{Cov}_{xv|0} + t^2\sigma_v^2}} \tag{103}$$

$$= \frac{\text{Cov}_{xv|0} + t\sigma_v^2}{\sigma_v \sigma_{x|0} \sqrt{1 + (2\,t\,\text{Cov}_{xv|0} + t^2\sigma_v^2)/\sigma_{x|0}^2}} \tag{104}$$

$$\backsim \frac{\text{Cov}_{xv|0} + t\sigma_v^2}{\sigma_v \sigma_{x|0}} \left[1 - \frac{1}{2}(2\,t\,\text{Cov}_{xv|0} + t^2\sigma_v^2)/\sigma_{x|0}^2 + \frac{3}{8}(2\,t\,\text{Cov}_{xv|0} + t^2\sigma_v^2)^2/\sigma_{x|0}^4 \quad \cdots \right] \tag{105}$$

$$= \frac{\text{Cov}_{xv|0}}{\sigma_v \sigma_{x|0}} + t\frac{\sigma_v^2 - \text{Cov}_{xv|0}^2}{\sigma_v \sigma_{x|0}} \quad \cdots \tag{106}$$

$$= \rho_{xv|0} + t\frac{\sigma_v^2 - \text{Cov}_{xv|0}^2}{\sigma_v \sigma_{x|0} \sigma_{x|0}^2} \quad \cdots \tag{107}$$

$$= \rho_{xv|0} + t\left(1 - \rho_{xv|0}^2\right)\frac{\sigma_v}{\sigma_{x|0}} \quad \cdots \tag{108}$$

3.5 Expansion and Contraction Time

We repeat here the equation for spread

$$\sigma_{x|t}^2 = \sigma_{x|0}^2 + 2\,t\,\text{Cov}_{xv|0} + t^2\sigma_v^2 \tag{109}$$

and emphasize that

- The spread has only a linear and quadratic term in time.
- The coefficient of the quadratic term is manifestly positive.
- This is not an approximation but is exact.

The quadratic term will eventually dominate since its coefficient is manifestly positive and hence the spread will always become infinite at infinite time. But the coefficient of the linear term may be negative and therefore the pulse may contract if the linear term dominates the quadratic term for a period of time. We now obtain that period of time. If we want smaller width than the original width, that is

$$\sigma_{x|t}^2 < \sigma_{x|0}^2 \tag{110}$$

then from Eq. (109) we have

$$\sigma_{x|0}^2 + 2\,t\,\text{Cov}_{xv|0} + t^2\sigma_v^2 < \sigma_{x|0}^2 \tag{111}$$

or

$$2\,t\,\mathrm{Cov}_{xv|0} + t^2\sigma_v^2 < 0 \tag{112}$$

giving that it will happen for contraction times, t_C, given by

$$0 \leq \, t_C \, \leq \, -2\,\frac{\mathrm{Cov}_{xv|0}}{\sigma_v^2} \tag{113}$$

Therefore for there to be contraction we must have

$$\mathrm{Cov}_{xv|0} < 0 \tag{114}$$

and why that should be the case will be seen simply when we explain our model. To obtain the time when the pulse achieves a minimum width, differentiate $\sigma_{x|t}^2$ with respect to time and set equal to zero,

$$\frac{\partial\sigma_{x|t}^2}{\partial t} = 2\,\mathrm{Cov}_{xv|0} + 2t\sigma_v^2 = 0 \tag{115}$$

yielding

$$t_M = \, -\frac{\mathrm{Cov}_{xv|0}}{\sigma_v^2} = \frac{1}{2}\,t_C \tag{116}$$

therefore the time to achieve the minimum contraction is equal to the time to go from the minimum contraction back to the original width. To determine how thin a pulse can get we substitute $t = t_M$ in Eq. (109) to obtain

$$\sigma_{x|t_M}^2 = \sigma_{x|0}^2\left(1 - \rho_{xv|0}^2\right) \tag{117}$$

where $\rho_{xv|0}$ is the correlation coefficient,

$$\rho_{xv|0}^2 = \frac{\mathrm{Cov}_{xv|0}^2}{\sigma_v^2\sigma_{x|0}^2} \tag{118}$$

We now give a physical picture explaining the above results. Consider the initial pulse to be a swarm of particles where each particle has a constant velocity and where the swarm has an average velocity V. We take V to be positive, that is the center of mass is moving to the right. Now suppose that initially the particles on the right have high velocity and the particles on the left have relatively slower velocity. Since everybody travels with a constant velocity the spread will be higher once the particles start to move. On the other hand suppose that fast particles are to the left and slow particles are to the right. Once the swarm starts to move the fast particles are catching up and the distance between the slow and fast particles is decreasing. That is the swarm contracts. Eventually the fast particles catch up and that is the time when the pulse width is minimum. Once they catch up, they pass them and the distance then continues to increase. The reason the covariance comes in such a fundamental way is that it tells us whether the fast particles are to the right or left of the pulse. If the covariance is positive the fast particles are to the right of the slow particles and if it is negative it means they are arranged the opposite way.

We now explain why the contraction time also depends on the spread of initial velocities. If the spread is very large then that means there is a big difference between the fast and slow particles and hence the spread in position will decrease faster since the fast particles can catch up quickly to the slow particles. Conversely if the velocity spread is small then it will take much longer for the fast particles to catch up to the slow particles.

Spread of Uncertainty Product. Using the fact that for pulse propagation

$$\sigma_{k|t}^2 = \sigma_{k|0}^2 \tag{119}$$

the uncertainty product evolves as

$$\begin{aligned} \sigma_{x|t}^2 \sigma_{k|t}^2 &= [\sigma_{x|t}^2 + 2t\mathrm{Cov}_{xv} + t^2\sigma_v^2]\sigma_{k|0}^2 &(120)\\ &= \sigma_{x|0}^2\sigma_{k|0}^2 + [2t\mathrm{Cov}_{xv} + t^2\sigma_v^2]\sigma_{k|0}^2 &(121)\end{aligned}$$

The uncertainty product eventually goes to infinity but can also momentarily decrease. When the uncertainty product is large the signal is said to be asymptotic. Thus in pulse propagation all signals become asymptotic.

4 Local Model of Pulse Propagation

We now give the details of a simple physical model that gives the identical equations as pulse propagation and from which all the quantities can be understood simply. More importantly we argue that the model can be used to visualize and understand pulse propagation in general. Motion of objects is simpler to visualize and understand than waves and certainly much of the vocabulary of waves is derived from the desire to make believe waves act like particles. We present a simple model of particle motion that is very close to wave motion and which hopefully will help to understand wave motion phenomena. The reason one may want to develop a particle view of wave motion is manifold. First, it may give us a clear and more intuitive picture of what is going on with wave phenomena. Second, it could lead to practical numerical schemes because instead of solving the wave equation we can evolve particles. Third, it may help to understand non linear waves which are particularly difficult to visualize. In addition, as we will show, in trying to understand wave propagation one comes across certain physical quantities that appear, mathematically, like "correlation", although their mathematical properties do not fully match those of classical correlations. By making a particle model we may be able to understand better correlations between quantities as they appear in the study of wave propagation.

Consider the simplest motion of a particle, the motion with a constant velocity,

$$x_t = x_0 + v\,t \tag{122}$$

where x_t is the position at time t, and the constant velocity is V. Now suppose we have a group of particles and the group is characterized by a density at the initial time. If all the particles have the same velocity it is clear that the group will move but keep its shape. Now suppose the velocity of each particle is not the same and suppose we pick the velocity of each particle depending on the initial position, x_0. Hence we write

$$v = v(x_0) \tag{123}$$

and now we have

$$x(t) = x_0 + v(x_0)\,t \tag{124}$$

For the density of particles we will use $\rho(x,t)$, that is

$$\rho(x,t) = \text{the density of particles at time } t \tag{125}$$

Also for convenience we use the following notation for the density at time zero,

$$\rho(x_0) = \rho(x,0) \tag{126}$$

It will be helpful to think of this common situation. Suppose we have a large number of runners at the starting line of a marathon and let us assume they take up a city block. We could place the runners in many different initial configurations. The most common arrangement is where one places the fast runners at the starting line, but we could place them at the end of the block, or in the middle, or any way we want. Each of these different ways is characterized by the function $v(x_0)$. How will the group of runners behave once the gun goes off? Clearly if the fast runners are at the starting line the swarm will immediately spread. But if the fast runners are at the back at the start of the race, then the swarm will first contract and then expand.

Now let us study how the group of runners behaves after the starting gun. As usual we define moments as

$$\langle\, x^n \,\rangle_t = \int x^n \rho(x,t) dx \tag{127}$$

However we do not have to know $\rho(x,t)$ explicitly because we can calculate everything from the initial density

$$\langle\, x^n \,\rangle_t = \int (x_0 + v(x_0)\,t)^n \rho(x_0) dx \tag{128}$$

First consider the center of mass. One takes expectation values of both sides of Eq. (124)

$$\langle\, x_t \,\rangle = \langle\, x_0 \,\rangle + \langle\, v(x_0) \,\rangle\, t \tag{129}$$

or

$$\langle\, x \,\rangle_t = \langle\, x \,\rangle_0 + Vt \tag{130}$$

where

$$V = \langle\, v_0 \,\rangle = \int v(x_0)\rho(x_0)dx_0 \tag{131}$$

Eq. (131) shows that the center of mass moves with a constant velocity which is the average velocity of the group at the initial time. Now consider the second moment. Square Eq. (124)

$$x_t^2 = (x_0 + v_0 t)^2 = x_0^2 + 2x_0 v_0 t + v_0^2 t^2 \tag{132}$$

and take averages of both sides, giving

$$\langle\, x^2 \,\rangle_t = \langle\, x^2 \,\rangle_0 + 2\langle\, xv \,\rangle_0 t + \langle\, v^2 \,\rangle_0 t^2 \tag{133}$$

where

$$\langle\, x_0 v_0 \,\rangle = \int x_0 v(x_0) P(x_0, 0) dx_0 \tag{134}$$

The spread of the swarm is

$$\sigma_{x|t}^2 = \langle\, x \,\rangle_t^2 - \langle\, x_t \,\rangle^2 \tag{135}$$

$$= \langle x^2\rangle_0 + 2\langle xv\rangle_0\, t + \langle v^2\rangle_0\, t^2 - (\langle x\rangle_0 + \langle v\rangle_0 t)^2 \tag{136}$$

$$= \langle x^2\rangle_0 - \langle x\rangle_0^2 + 2(\langle xv\rangle_0 \; - \langle x\rangle_0\langle v\rangle_0)t + (\langle v^2\rangle_0 \; - \langle v\rangle_0^2)t^2 \tag{137}$$

or,

$$\sigma_{x|t}^2 = \sigma_{x|0}^2 + 2\,\mathrm{Cov}_{vx|0} t + \sigma_{v|0}^2 t^2 \tag{138}$$

where $\sigma_{v|0}^2$ is the standard deviation of the initial velocity

$$\sigma_{v|0}^2 = \langle v^2\rangle_0 - \langle v\rangle_0^2 \tag{139}$$

and where the covariance is

$$\mathrm{Cov}_{vx|0} = \langle xv(x)\rangle_0 - \langle v\rangle_0\langle x\rangle_0 \tag{140}$$

It is the covariance between the initial position and initial velocity. One can write the spread in terms of the correlation coefficient,

$$\sigma_{x|t}^2 = \sigma_{x|0}^2 + 2\rho_{vx|0}\sigma_{x|0}\sigma_{v|0} t + \sigma_{v|0}^2 t^2 \tag{141}$$

where the correlation coefficient is

$$\rho_{vx|0} = \frac{\mathrm{Cov}_{xv|0}}{\sigma_{x|0}\,\sigma_{v|0}} \tag{142}$$

With one exception, the coefficient of t^2 in Eq. (141) can never be made zero and therefore the swarm must spread to infinity. The coefficient of t^2 can be made zero only if we take $v_0 = c$. For that case we have that $\mathrm{Cov}_{xv|t}$ and $\sigma^2_{v|0}$ are both zero and hence $\sigma^2_{x|t} = \sigma^2_{x|0}$ and the swarm will not spread. While it is certain that the swarm will spread *eventually,* the swarm can contract for a certain time. For the pulse to contract we must have $\mathrm{Cov}_{xv} \leq 0$ which gives

$$0 \leq \, t \leq \, -2\,\frac{\mathrm{Cov}_{xv|0}}{\sigma^2_{v|0}} \tag{143}$$

for the contraction time.

Let us also calculate the covariance as a function of time. First, we have that

$$x(t)v(t) = (x_0 + v_0 t)v_0 = x_0 v_0 + v_0^2\, t \tag{144}$$

Taking averages we obtain

$$\langle xv\rangle_t = \langle xv\rangle_0 + \langle\, v_0^2 \,\rangle\, t \tag{145}$$

The covariance at time t is then

$$\mathrm{Cov}_{vx|t} = \langle xv\rangle_t - \langle x\rangle_t \langle v\rangle_t \tag{146}$$

which gives

$$\mathrm{Cov}_{xv|t} \,= \mathrm{Cov}_{xv|0} + \sigma^2_{v|0}\, t \tag{147}$$

We see that the covariance must increase. For the correlation coefficient we have

$$\rho_{xv|t} = \frac{\mathrm{Cov}_{xv|t}}{\sigma_{x|t}\sigma_{v|t}} = \frac{\mathrm{Cov}_{xv|0} + \sigma^2_{v|0} t}{\sigma_{v|0}\,\sqrt{\sigma^2_{x|0} + 2\,\mathrm{Cov}_{vx|0} t + \sigma^2_{v|0} t^2}} \tag{148}$$

The covariance and correlation coefficient increase because as the swarm spreads the fast particles will be more and more up front of the swarm and will hence be highly correlated with the relative position in the swarm. Indeed as time goes to infinity the correlation coefficient goes to 1,

$$\rho_{xv|t} \to 1 \qquad \text{as} \qquad t \to \infty \tag{149}$$

To understand these equations consider the case where the fast runners are lined up in the front. The initial covariance is positive, which indicates that fact. At the start the fast runners take off, the slow runners at the back do not keep up and the swarm immediately spreads. However, suppose that the fast runners are in the back: the covariance is then negative which means that the slow runners are up front. Now, for a short time after time at the start of the race the fast runners are catching up to the slow runners and therefore the swarm contracts. Also, the bigger the initial difference between the fast and slow runners, the longer will be the contraction time. In addition the bigger the difference in speed between the fast and slow runners the smaller will be the contraction time, because the fast runners will catch up more quickly. The same type of behavior is exhibited by pulse propagation in dispersive media.

5 Instantaneous Frequency and Dispersion

"Instantaneous frequency" is one of the most fundamental quantities; it is frequency as a function of time. While it is intuitively obvious that frequencies do change, its exact mathematical expression is far from obvious. Generally speaking instantaneous frequency is the derivative of the phase and the question has been how do we determine the phase. This is an old and important problem and the general solution is usually the one given by Gabor. Take the real signal and delete its negative frequencies and form a new signal just from the positive ones. That results in a complex signal and the derivative of the phase is then the instantaneous frequency. However this procedure has not been fully investigated for a pulse and we do not do so here but just point this out. Here we use the solution to the wave equation and express it in terms of amplitude and phase,

$$u(x,t) = A(x,t)e^{i\varphi(x,t)} \tag{150}$$

to define instantaneous frequency as

$$\omega_i(x,t) = -\frac{\partial}{\partial t}\varphi(x,t) \tag{151}$$

and spatial local frequency by

$$k_i(x,t) = \frac{\partial}{\partial t}\varphi(x,t) \tag{152}$$

The reason for the differences in sign is because of the basic definition, Eq. (2).

We now address the problem as to how does the instantaneous frequency vary from point to point as a pulse propagates and how does dispersion effect that. For example suppose a sound is made and there are two people each standing at 50 and 100 feet away. Will they hear the same sound? The answer is yes because for frequencies that humans speak at there is no dispersion in air. However suppose there is dispersion, how would the frequencies change? That is how does the instantaneous frequency of a pulse depend from position to position? We have not been able to do this exactly but we have done it approximately using the asymptotic solution. However, for many examples we have been able to do it exactly and hence we can compare our approximate solution with these exact solutions. For convenience we repeat here the asymptomatic solution

$$u_a(x,t) \sim S(k_s,0)\sqrt{\frac{1}{tW''(k_s)}}e^{ik_sx-iW(k_s)t-i\pi\mathrm{sgn}\,W''/4} \tag{153}$$

and the amplitude and phase are

$$A_a(x,t) = |S(k_s,0)|\sqrt{\frac{1}{tW''(k_s)}} \tag{154}$$

$$\varphi_a(x,t) = \psi(k_s,0) + k_sx - W(k_s)t - \pi\mathrm{sgn}\,W''/4 \tag{155}$$

where the value of k_s is obtained from solving

$$W'(k_s) = x/t \tag{156}$$

Differentiating the phase, $\varphi_a(x,t)$, we have

$$\begin{aligned}
\omega_i(x,t) &= -\frac{\partial}{\partial t}\varphi_a(x,t) && (157)\\
&= \frac{d\psi}{dk_s}\frac{\partial k_s}{\partial t} + x\frac{\partial k_s}{\partial t} - t\frac{dW(k_s)}{dk_s}\frac{\partial k_s}{\partial t} + W(k_s) && (158)\\
&= -\left[\frac{d\psi}{dk_s} + x - t\frac{dW(k_s)}{dk_s}\right]\frac{\partial k_s}{\partial t} + W(k_s) && (159)\\
&= -\left[\frac{d\psi}{dk_s} + x - t\frac{x}{t}\right]\frac{\partial k_s}{\partial t} + W(k_s) && (160)
\end{aligned}$$

and hence

$$\omega_i(x,t) = -\frac{d\psi}{dk_s}\frac{\partial k_s}{\partial t} + W(k_s) \tag{161}$$

From Eq. (156) we have

$$W''(k_s)\frac{\partial k_s}{\partial t} = -x/t^2 \tag{162}$$

giving

$$\begin{aligned}
\frac{\partial k_s}{\partial t} &= -\frac{x}{t^2W''(k_s)} && (163)\\
&= -\frac{W'(k_s)}{tW''(k_s)} && (164)\\
&= -\frac{W^2(k_s)}{xW''(k_s)} && (165)
\end{aligned}$$

and therefore we can express the instantaneous frequency in the asymptotic regime in a variety of ways,

$$\omega_i(x,t) = \frac{x}{t^2 W''(k_s)} \frac{d\psi}{dk_s} + W(k_s) \tag{166}$$

$$= \frac{W'(k_s)}{tW''(k_s)} \frac{d\psi}{dk_s} + W(k_s) \tag{167}$$

$$= \frac{W'^2(k_s)}{xW''(k_s)} \frac{d\psi}{dk_s} + W(k_s) \tag{168}$$

It should not be concluded from these expressions that the instantaneous frequency varies in any specific way. For example by looking at Eq. (166) one may think that is a linear function of x. That would not be right because in all of the above examples k_s is a function of x and t. However one can get a sense of that if we expand the dispersion relation in a power series.

Spatial Local Frequency. We now consider the issue of spatial instantaneous frequency. We define it by

$$k_i(x,t) = \frac{\partial}{\partial x}\varphi(x,t) \tag{169}$$

Differentiating the phase, $\varphi_a(x,t)$, we have

$$k_i(x,t) = \frac{\partial}{\partial x}\varphi_a(x,t) \tag{170}$$

$$= \frac{d\psi}{dk_s}\frac{\partial k_s}{\partial x} + x\frac{\partial k_s}{\partial x} + k_s - t\frac{dW(k_s)}{dk_s}\frac{\partial k_s}{\partial x} \tag{171}$$

$$= \left[\frac{d\psi}{dk_s} + x - t\frac{dW(k_s)}{dk_s}\right]\frac{\partial k_s}{\partial x} + k_s \tag{172}$$

$$= \left[\frac{d\psi}{dk_s} + x - t\frac{x}{t}\right]\frac{\partial k_s}{\partial x} + k_s \tag{173}$$

$$= \frac{d\psi}{dk_s}\frac{\partial k_s}{\partial x} + k_s \tag{174}$$

But from Eq. (156) we now have that

$$W''(k_s)\frac{\partial k_s}{\partial x} = 1/t \tag{175}$$

giving

$$\frac{\partial k_s}{\partial x} = \frac{1}{tW''(k_s)} \tag{176}$$

$$= \frac{W'(k_s)}{xW''(k_s)} \tag{177}$$

and hence

$$k_i(x,t) = \frac{1}{tW''(k_s)}\frac{d\psi(k,0)}{dk_s} + k_s \tag{178}$$

$$= \frac{W'(k_s)}{xW''(k_s)}\frac{d\psi(k,0)}{dk_s} + k_s \tag{179}$$

This is the spatial instantaneous frequency.

Translation invariance. We point out that while the exact solution is translationly invariant the asymptotic solution is not. In particular suppose we define

$$u_{tr}(x,0) = u(x-x_0,0) \tag{180}$$

then for the exact solution

$$u_{tr}(x,t) = u(x-x_0,t) \tag{181}$$

Therefore, if we work things out for $u(x,t)$ it is easy to take all formulas and change them for the case $u(x-x_0,t)$. One merely substitutes $x-x_0$ for x. However for the asymptotic solution that is not the case, but we can obtain a formula to handle the situation. In general we have that

$$S_{tr}(k,t) = e^{ikx_0}S(k,t) \tag{182}$$

and therefore

$$u_{a,tr}(x,t) = e^{ik_sx_0}u_a(x,t) \tag{183}$$

The phase hence changes according to

$$\varphi_{a,tr}(x,t) = \varphi_a(x,t) + k_sx_0 \tag{184}$$

and the instantaneous frequency is therefore

$$\omega_{i,tr}(x,t) = \omega_i(x,t) - \frac{\partial k_s}{\partial t}x_0 \tag{185}$$

$$= \omega_i(x,t) + \frac{xx_0}{t^2W''(k_s)} \tag{186}$$

similarly

$$k_{i,tr}(x,t) = k_i(x,t) - \frac{\partial k_s}{\partial x}x_0 \tag{187}$$

$$= k_i(x,t) - \frac{x_0}{tW''(k_s)} \tag{188}$$

6 Higher Order Dispersion

Historically for a variety of reasons dispersion relation which are at most quadratic have been intensively studied. Perhaps the main reason is that for that case certain integrals can be carried out. Higher order dispersion is when there are terms in the dispersion relation that are higher than quadratic and it is important to understand their effect. Now in our previous development we have shown how to obtain exact relations for any dispersion relation and therefore we should be able to get exact results illustrating higher order dispersion. To obtain a sense of higher order dispersion we first do a specific example and then see if with the simple example there are qualitatively different results. If so then we will try to prove these in general. The example we use is where we add a cubic term to the dispersion relation [10]

$$W(k) = ck + \gamma k^2/2 + \eta k^3/3 \tag{189}$$

and for the initial pulse we take

$$u(x,0) = (\alpha/\pi)^{1/4}\, e^{-\alpha x^2/2 + i\beta x^2/2 + ik_0 x} \tag{190}$$

For each physical quantity of interest we shall give the result and discuss the effects of the cubic dispersion term.

Velocity of the center of mass. One obtains

$$V = c + \gamma k_0 + \eta \left[k_0^2 + \frac{1}{2\alpha'} \right] \tag{191}$$

$$= c + \gamma k_0 + \eta \left[k_0^2 + \frac{\alpha^2 + \beta^2}{2\alpha} \right] \tag{192}$$

Without the cubic term, it is seen that what enters are c and k_0, and that is reasonable because k_0 is just the average spatial frequency of the initial pulse. However, adding the cubic term introduces fundamental new parameters, namely α and β, which were absent before.

Spread of Group Velocity. The spread is

$$\sigma_v^2 = \frac{\gamma^2}{2\alpha'} + \frac{\gamma\eta k_0}{\alpha'} + \frac{\eta^2}{\alpha'} \left[2k_0^2 + \frac{1}{2\alpha'} \right] \tag{193}$$

$$= \frac{1}{\alpha'} \left(\frac{\gamma^2}{2} + \gamma\eta k_0 + \eta^2 \left[2k_0^2 + \frac{1}{2\alpha'} \right] \right) \tag{194}$$

$$= \frac{\alpha^2 + \beta^2}{\alpha} \left(\frac{\gamma^2}{2} + \gamma\eta k_0 + \eta^2 \left[2k_0^2 + \frac{1}{2\alpha'} \right] \right) \tag{195}$$

Covariance. As we discussed the sign of the covariance is crucial in determining whether a pulse will contract and for how long the contraction will go on, and therefore it is important to see how higher order dispersion effects that. The covariance is now given by

$$\mathrm{Cov}_{xv} = \frac{\beta}{\alpha}\left[\frac{\gamma}{2} + \eta k_0\right] \tag{196}$$

The higher order results now depends on k_0. If we only have the quadratic term then the sign of the covariance is only determined by the sign of β, assuming that both α and γ are positive. But now we have another possibility because the sign and magnitude of k_0 is at our disposal. In fact, we can make the covariance less than zero even if we take all other parameters positive. This will be the case when

$$\mathrm{Cov}_{xv} \leq 0 \qquad \text{for} \qquad k_0 \leq \frac{1}{\eta}\left[\frac{\alpha}{\beta} - \frac{\gamma}{2}\right] \tag{197}$$

Spread. Putting the above results into Eq. (67) we have that the spread is

$$\sigma_{x|t}^2 = \sigma_{x|0}^2 + 2t\left[\frac{\beta}{\alpha}\left(\frac{\gamma}{2} + \eta k_0\right)\right] + t^2\left[\frac{\gamma^2}{2\alpha'} + \frac{\gamma\eta k_0}{\alpha'} + \frac{\eta^2}{\alpha'}\left(2k_0^2 + \frac{1}{2\alpha'}\right)\right] \tag{198}$$

$$= \sigma_{x|0}^2 + 2t\left[\frac{\beta}{\alpha}\left(\frac{\gamma}{2} + \eta k_0\right)\right] + \frac{t^2}{\alpha'}\left[\frac{\gamma^2}{2} + \gamma\eta k_0 + \eta^2\left(2k_0^2 + \frac{1}{2\alpha'}\right)\right] \tag{199}$$

$$= \sigma_{x|0}^2\left[1 + 2\beta t(\gamma + 2\alpha\eta k_0) + (\alpha^2 + \beta^2)t^2\left\{\gamma^2 + 2\gamma\eta k_0 2\eta^2\left(2k_0^2 + \frac{\alpha^2 + \beta^2}{2\alpha}\right)\right\}\right] \tag{200}$$

Contraction. Using Eq. (198) the pulse will contract for

$$0 \leq t \leq T \tag{201}$$

where

$$T = -\frac{\frac{\beta\alpha'}{\alpha}\left(\frac{\gamma}{2} + \eta k_0\right)}{\gamma^2/2 + \gamma\eta k_0 + \eta^2\left(2k_0^2 + \frac{1}{2\alpha'}\right)} \tag{202}$$

$$= -\frac{2\beta}{\gamma(\alpha^2 + \beta^2)}\left[1 + \frac{\eta^2\left(2k_0^2 + \frac{1}{2\alpha'}\right)}{(1 + 2\eta k_0)}\right]^{-1} \tag{203}$$

If one compares the cubic case with the quadratic case we see that an important new signal parameter comes into play, namely k_0. It is probably the case that this parameter would also enter into the quadratic case if an initial pulse was taken so that it would not be symmetrical about k_0. Nonetheless what our example shows is that there is a wide variety of possibilities to control the contraction and rate of expansion of a pulse. It is important to appreciate that the contrasts may just be due to the fact that we have used a Gaussian example and that perhaps for the quadratic term the peculiarities arise just because of that. That is, perhaps the reason the cubic terms effects arise is artificial, and perhaps they really are there in general but absent from this examples. Further studies are needed to understand this fully but what the above shows is one can calculate appropriate quantities exactly for the sake of comparison.

7 Wigner Distribution Approach

We now discuss how the Wigner distribution [18, 3, 4] gives considerable insight and calculational advantage into pulse propagation. It is natural that it should be the case since the aim of time-frequency analysis is to show how frequencies change in time. The spatial/spatial-frequency Wigner distribution is [11, 15]

$$W(x,k,t)=\frac{1}{2\pi}\int u^*(x-\tfrac{1}{2}\tau,t)\,u(x+\tfrac{1}{2}\tau,t)\,e^{-i\tau k}\,d\tau \tag{204}$$

and in terms of the spatial spectrum it is

$$W(x,k,t)=\frac{1}{2\pi}\int S^*(k+\tfrac{1}{2}\theta,t)\,S(k-\tfrac{1}{2}\theta,t)\,e^{-i\theta x}\,d\theta \tag{205}$$

We first give the important moments in a general way, that is without specifying that we are dealing with pulse propagation. Before giving these moments we repeat here the basic definition of phases and amplitudes for both the signal and spectrum

$$u(x,t)=A(x,t)e^{i\varphi(x,t)} \tag{206}$$

$$S(k,t)=\frac{1}{\sqrt{2\pi}}\int u(x,t)\;e^{-ikx}\,dx \tag{207}$$

$$=B(k,t)e^{i\psi(k,t)} \tag{208}$$

The Wigner distribution satisfies the marginals,

$$\int W(x,k,t)dk=|\,u(x,t)|^2 \tag{209}$$

$$\int W(x,k,t)dx=|\,S(k,t)\,|^2 \tag{210}$$

For the sake of clarity in this section we will not normalize the moments with their respective marginals.

A straightforward calculation yields

$$\langle k\rangle_{x,t}=\int kW(x,k,t)dk \tag{211}$$

$$=\frac{\partial}{\partial x}\varphi(x,t) \tag{212}$$

which is the instantaneous spatial frequency. Also,

$$\langle x\rangle_{k,t}=B(k,t)\int xW(x,k,t)dx \tag{213}$$

$$=-\frac{\partial}{\partial k}\psi(k,t) \tag{214}$$

and the covariance is

$$\langle kx\rangle_t = \int xkW(x,k,t)dxdk \tag{215}$$

$$= \int x\frac{\partial\varphi(x,t)}{\partial x}|u(x,t)|^2 dx \tag{216}$$

$$= -\int k\frac{\partial\psi(k,t)}{\partial k}|S(k,t)\,|^2 \tag{217}$$

Now let us specialize to the case of pulse propagation. We repeat here for convenience the basic relevant equations that characterize pulse propagation

$$S(k,t) = S(k,0)\ e^{-iW(k)t} \tag{218}$$

$$= B(k,t)e^{i\psi(k,t)} \tag{219}$$

$$= B(k,0)e^{i\psi(k,0)-iW(k)t} \tag{220}$$

and

$$B(k,t) = B(k,0) \tag{221}$$

$$\psi(k,t) = \psi(k,0) - W(k)t \tag{222}$$

Therefore using Eq. (222) we have

$$\langle x\rangle_{k,t} = -\frac{\partial}{\partial k}\psi(k,t) \tag{223}$$

$$= -\frac{\partial}{\partial k}\psi(k,0) + W'(k)t \tag{224}$$

$$= -\frac{\partial}{\partial k}\psi(k,0) + v(k)t \tag{225}$$

But at $t = 0$

$$\langle x\rangle_{k,0} = -\frac{\partial}{\partial k}\psi(k,0) \tag{226}$$

and therefore

$$\langle x\rangle_{k,t} = \langle x\rangle_{k,0} + v(k)t \tag{227}$$

This is an interesting result because it predicts exactly the picture we developed previously. Now consider

$$\langle kx\rangle_t = -\int k\frac{\partial\psi(k,t)}{\partial k}|S(k,t)\,|^2 \tag{228}$$

$$= -\int k(\frac{\partial}{\partial k}\psi(k,0) - v'(k)t)|S(k,0)\,|^2 \tag{229}$$

$$= \langle kx\rangle_0 + \langle kv(k)\rangle_0 t \tag{230}$$

which agrees with our previous results. The important point to observe here is that these expressions and results came out naturally using the Wigner distribution.

Evolution of the Wigner distribution and an approximation. Our aim is to express the Wigner distribution at time t in terms of the Wigner distribution at time $t = 0$. The reason we consider this problem is because a considerable simplification occurs. For pulse propagation we substitute Eq. (218) into Eq. (205)

$$W(x,k,t) = \frac{1}{2\pi}\int S^*(k+\tfrac{1}{2}\theta,0)\,S(k-\theta/2,0)\,e^{-i\theta x}e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t}\,d\theta \tag{231}$$

Starting with

$$W(x,k,0) = \frac{1}{2\pi}\int S^*(k+\tfrac{1}{2}\theta,0)\,S(k-\tfrac{1}{2}\theta,0)\,e^{-i\theta x}\,d\theta \tag{232}$$

we invert to obtain,

$$S^*(k+\tfrac{1}{2}\theta,0)\,S(k-\tfrac{1}{2}\theta,0) = \int W(x,k,0)\,e^{i\theta x}\,dx \tag{233}$$

and inserting into Eq. (231) we have

$$W(x,k,t) = \frac{1}{2\pi}\iint W(x',k,0)\,e^{-i\theta(x'-x)}e^{i[W(k+\theta/2)-W(k-\theta/2)]t}\,d\theta\,dx' \tag{234}$$

This is exact. It is convenient to write this in terms of a Green's function for the Wigner distribution,

$$W(x,k,t) = \frac{1}{2\pi}\int W(x',k,0)L(x'-x,t)\,dx' \tag{235}$$

where

$$L(x'-x,k,t) = \int\ e^{-i\theta(x'-x)}e^{i[W(k+\theta/2)-W(k-\theta/2)]t}\,d\theta \tag{236}$$

It is very interesting to find an approximation to the Wigner distribution with the following aim. The evolution may be simple because we are in phase space and it is generally the case that evolution equations take on a particularly simple form in comparison to evolution of the density itself. Expand $[W(k+\theta/2)-W(k-\theta/2)]$ in θ

$$W(k+\theta/2)-W(k-\theta/2) = \sum_{n=0}^{\infty}\frac{W^{(2n+1)}(k)}{(2n+1)!}\frac{\theta^{2n+1}}{2^n} \sim v(k)\theta + \frac{1}{24}v^{(2)}(k)\theta^3 \quad \cdots \tag{237}$$

where $v^{(2n+1)}(k)$ is the $2n+1$ derivative with respect to k. Keeping only the first term

$$L(x'-x,t) \sim \int\ e^{-i\theta(x'-x)}e^{iv(k)t\theta}\,d\theta = \delta(x'-x+v(k)t) \tag{238}$$

and substituting into Eq. (119) we have

$$W(x,k,t) \sim W(x - v(k)t, k, 0) \tag{239}$$

This is a remarkably simple result and gives a very simple way of propagating the Wigner distribution for pulses. Note that no calculations are required, just a simple substitution. That is, no equations have to be solved or integrated. This method may be a good practical method for evolving pulses in time. We now address how the method can be improved. What is needed is further approximation by keeping more and more successive terms in the expansion given by Eq. (123). Going to the next term we have

$$L(x' - x, t) \sim \int e^{-i\theta(x'-x)} e^{iv(k)t\theta + iv''(k)t\theta^3/24}\, d\theta \tag{240}$$

$$= \int e^{-i\theta[(x'-x-v(k)t]} e^{iv''(k)t\theta^3/24}\, d\theta \tag{241}$$

This integral can be expressed in terms of Airy functions but we do not do so here. This may offer a considerable better approximation than Eq. (239)

We point out that the recovery of the pulse from the Wigner distribution is an important problem. The signal can be recovered from the Wigner distribution up to a constant phase factor. However in this case the constant phase factor may be a function of time! This issue has been considered by Leavens and Mayato.

Example. Consider the initial pulse given by

$$u(x,0) = (\alpha/\pi)^{1/4}\, e^{-\alpha x^2/2 + i\beta x^2/2 + ik_0 x} \tag{242}$$

The Wigner distribution is

$$W(x,k,0) \; = \; \frac{1}{\pi}\, e^{-\alpha x^2 - (k - \beta x - k_0)^2/\alpha} \tag{243}$$

Therefore we have

$$W(x,k,t) \sim W(x - v(k)t, k, 0) \tag{244}$$

$$= \frac{1}{\pi}\, e^{-\alpha(x - v(k)t)^2 - (k - k_0 - \beta(x - v(k)t))^2/\alpha} \tag{245}$$

This is a general result for the initial pulse where the dispersion relation is arbitrary. If we now chose

$$W(k) = ck + \gamma k^2/2 \tag{246}$$

$$v(k) = c + \gamma k \tag{247}$$

we have

$$
\begin{aligned}
W(x,k,t) &\sim W(x-v(k)t,k,0) && (248)\\
&= W(x-(c+k\gamma t),k,0) && (249)\\
&= \frac{1}{\pi}\exp\left[-\alpha((x-c-k\gamma t)^2-(k-\beta(x-c-k\gamma t-k_0)^2/\alpha\right] && (250)
\end{aligned}
$$

Explicitly

$$
W_a(x,k,t) \sim \frac{1}{\pi}\exp\left[-\alpha((x-c-k\gamma t)^2-(k-\beta(x-c-k\gamma t-k_0)^2/\alpha\right] \qquad (251)
$$

It is remarkable that for this example the answer is exact!

A further example is

$$
W(k) = ck+\gamma k^3/3 \qquad ; \qquad v(k)=c+\gamma k^2 \qquad (252)
$$

and hence

$$
\begin{aligned}
W_a(x,k,t) &= W(x-v(k)t,k,0) && (253)\\
&= W(x-ct-\gamma k^2 t,k,0) && (254)\\
&= \frac{1}{\pi}\,e^{-\alpha((x-ct-\gamma k^2t)^2-(k-k_0-\beta(x-ct-\gamma k^2t)^2/\alpha} && (255)
\end{aligned}
$$

8 Case B

We now address the case where we are at a fixed position and generate a pulse, which is the situation appropriate in radar, sonar and fiber optics, etc. We generate $u(0,t)$ where we have taken $x=0$ where the pulse is being generated. Many new concepts are now introduced for this case. An important issue is whether we stick with the definitions of the relation of pulse and spectrum as before or we change to make it easier and more conventional to standard signal analysis. We have decided to take

$$
e^{-ikx+i\omega t} \qquad (256)
$$

for the fundamental solution. Substituting into the wave equation

$$
\sum_{n=0}^{N} a_n\frac{\partial^n u}{\partial t^n} = \sum_{n=0}^{M} b_n\frac{\partial^n u}{\partial x^n} \qquad (257)
$$

one obtains

$$
\sum_{n=0}^{N} a_n(i\omega)^n - \sum_{n=0}^{M} b_n(-ik)^n = 0 \qquad (258)
$$

Solve for k in terms of ω

$$k = K(\omega) \tag{259}$$

The general solution is then given by

$$u(x,t) \;=\; \frac{1}{\sqrt{2\pi}} \int F(\omega,x)\, e^{i\omega t}\, d\omega \tag{260}$$

with

$$F(\omega,x) = F(\omega,0)\, e^{-iK(\omega)x} \tag{261}$$

where $F(\omega,0)$ is the spectrum evaluated at $x=0$,

$$F(\omega,0) = \frac{1}{\sqrt{2\pi}} \int u(0,t)\; e^{-i\omega t}\, dt \tag{262}$$

Also,

$$F(\omega,x) \;=\; \frac{1}{\sqrt{2\pi}} \int u(x,t)\, e^{-i\omega t}\, dt \tag{263}$$

That is $u(x,t)$ and $F(\omega,x)$ form Fourier transform pairs for any x. In analogy with group velocity, $K'(\omega)$, will be of importance . In particular we define the frequency transit $\tau(\omega)$ for a given frequency by

$$\tau(\omega) = K'(\omega) \tag{264}$$

We call it the frequency transit time because as we will see it will turn out to be the time taken for a given frequency to travel a unit distance. We now discuss some important concepts for this case and ones thinking must change from the previous case. One must visualize that one is staying put at a particular place and is measuring a number of important quantities. Suppose we have a signal given by $f(t)$, the mean time is traditionally defined by

$$\langle\, t\,\rangle = \int t\,|f(t)|^2\, dt \tag{265}$$

The only difference between this and the usual definition is that now will be a function of position. Similarly the duration of the signal is given by

$$\sigma^2 = \int (t \,-\, \langle\, t\,\rangle)^2 |f(t)|^2\, dt \tag{266}$$

In the case of pulse propagation we use the same definitions but this signal will be $u(x,t)$ and hence is a function of position. We now develop the analogous ideas and formulas for this case as we did before and will introduce a number of new and interesting viewpoints.

We first developed the general approach in terms of calculating moments, that is moments at a particular point in space,

$$\langle\, t^n \,\rangle_x = \int t^n \, |u(x,t)|^2 \, dt \tag{267}$$

$$= \int F^*(\omega, x) \mathcal{T}^n F(\omega, x) \, d\omega \tag{268}$$

where $\mathcal{T}$ is the time operator in the frequency domain

$$\mathcal{T} = i \, \frac{\partial}{\partial \omega} \tag{269}$$

We now discuss each moment and its physical interpretation. Consider the first moment. It is given by

$$\langle\, t \,\rangle_x = \int t \, |u(x,t)|^2 \, dt \tag{270}$$

$$= \int F^*(\omega, x) \mathcal{T} F(\omega, x) \, d\omega \tag{271}$$

and this evaluates to

$$\langle\, t \,\rangle_x = \langle\, t \,\rangle_0 + Tx \tag{272}$$

where

$$T = \int \tau(\omega) \, |F(\omega, 0)|^2 \, d\omega \tag{273}$$

These equations lead to the following interpretation. $\tau(\omega)$ is the time that it takes a sine wave generated with frequency ω to travel a unit distance and T is the average time for all frequencies where the spectrum is used as the weighting function. Therefore the mean time at position x, $\langle\, t \,\rangle_x$, is the mean time at position $\langle\, t \,\rangle_0$ plus the time it took to travel from the initial position to x. Now that quantity is Tx because as just discussed T is the time per unit distance and x is the distance traveled since the particle started at $x = 0$ initially.

For the second moment

$$\langle\, t^2 \,\rangle_x = \langle t^2 \rangle_0 + x \langle \tau \mathcal{T} + \mathcal{T} \tau) \rangle_0 + x^2 \langle \tau^2 \rangle_0 \tag{274}$$

where

$$\langle \tau \mathcal{T} + \mathcal{T} \tau) \rangle_0 \tag{275}$$

Where

$$\langle \tau \mathcal{T} + \mathcal{T} \tau) \rangle_x = \int F^*(\omega, x) \left[\tau(\omega) \mathcal{T} + \mathcal{T} \tau(\omega) \right] \, F(\omega, x) d\omega \tag{276}$$

$$= \int F^*(\omega, x) \left[\tau(\omega) v(k) i \frac{\partial}{\partial \omega} + i \frac{\partial}{\partial \omega} \tau(\omega) v(k) \right] \, F(\omega, x) d\omega \tag{277}$$

$$= i \int F^*(\omega, x) \left[2\tau(\omega) \frac{\partial F(k,t)}{\partial \omega} + \tau'(\omega) F(k,t) \right] \, d\omega \tag{278}$$

A standard notation is

$$[\tau(\omega), t]_+ = \tau(\omega)\mathcal{T} + \mathcal{T}\tau(\omega) \tag{279}$$

Also if we define

$$\mathcal{W} = -i\frac{\partial}{\partial t} \tag{280}$$

then

$$\langle \tau(\mathcal{W})t + t\tau(\mathcal{W})\rangle_x = \int u^*(x,t)\ [\tau(\mathcal{W})t + t\tau(\mathcal{W})]\ u(x,t)\,dt \tag{281}$$

The duration is

$$\sigma^2_{t|x} = \sigma^2_{t|0} + 2\,x\,\mathrm{Cov}_{t\tau} + x^2\sigma^2_\tau \tag{282}$$

where

$$\sigma^2_\tau = \int (\tau(\omega) - T)^2\,|F(\omega,0)|^2\,d\omega \tag{283}$$

$$\mathrm{Cov}_{t\tau} = \langle\,\tau t\,\rangle_x - \langle\,\tau\,\rangle_0\langle\,t\,\rangle_0 \tag{284}$$

The mathematics is as before and if we write

$$F(\omega,x) = |F(\omega,x)|e^{i\psi(\omega,x)} \tag{285}$$

then in general

$$\langle\,\tau t\,\rangle_x = \tfrac{1}{2}\langle\,\tau\mathcal{T} + \mathcal{T}\tau\,\rangle_x \tag{286}$$

$$= -\int \tau(\omega)\frac{\partial\psi}{\partial\omega}|F(\omega,x)|^2\,d\omega \tag{287}$$

Now let us consider how duration changes with position. As before we note the dominant term is $x^2\sigma^2_\tau$ (for large x) and its coefficient is manifestly positive. Therefore for large distances the duration must go to infinity no matter what the duration is at the point where it is generated. That is at large distances, and in particular at infinity the duration is infinite.

We now calculate at what spatial points x_C the duration is smaller than the duration at the point of generation, $x = 0$. The duration will be shorter if $\sigma^2_{t|x} < \sigma^2_{t|0}$, which gives

$$0 \leq\ x_C \leq\ -2\,\frac{\mathrm{Cov}^2_{t\tau|0}}{\sigma^2_\tau} \tag{288}$$

It will be a minimum at the following point

$$x_M = -\frac{\mathrm{Cov}_{t\tau|0}}{\sigma^2_\tau} = \frac{1}{2}\,x_C \tag{289}$$

and the duration at x_M is

$$\sigma^2_{t|x_M} = \sigma^2_{t|0} - \frac{\mathrm{Cov}_{t\tau|x}}{\sigma^2_\tau} = \sigma^2_{t|0}\left(1 - \rho_{t\tau|x}\right) \tag{290}$$

where

$$\rho^2_{t\tau|0} = \frac{\mathrm{Cov}^2_{t\tau|0}}{\sigma_t^2 \sigma^2_{\tau|0}} \tag{291}$$

The Covariance Between Time and Transit Time. As before the covariance is the crucial quantity, but now it is the covariance between time and frequency transit time

$$\mathrm{Cov}_{t\tau|x} = \langle\, t\tau \,\rangle_x - \langle\, \tau \,\rangle_x \langle\, t \,\rangle_x \tag{292}$$

If we express the spectrum in terms of amplitude and phase then

$$F(\omega, x) = |F(\omega, x)|\, e^{i\psi(\omega,x)} \tag{293}$$

$$= F(\omega, 0)\, e^{-iK(\omega)x} \tag{294}$$

$$= |F(\omega, 0)|\, e^{i\psi(\omega,0)} e^{-iK(\omega)x} \tag{295}$$

and therefore the phase as a function of position is

$$\psi(\omega, x) = \psi(\omega, 0) - K(\omega)x \tag{296}$$

and hence

$$\frac{\partial\psi(\omega, x)}{\partial\omega} = \frac{\partial\psi(\omega, 0)}{\partial\omega} - K'(\omega)x \tag{297}$$

or

$$\frac{\partial\psi(\omega, x)}{\partial\omega} = \frac{\partial\psi(\omega, 0)}{\partial\omega} - \tau x \tag{298}$$

Substituting this into Eq. (287) we have

$$\langle t\, \tau \,\rangle_x = \langle t\, \tau \,\rangle_0 + \langle\, \tau^2 \,\rangle x \tag{299}$$

Therefore

$$\mathrm{Cov}_{t\tau|x} = \langle t\tau \,\rangle_x - \langle\, \tau \,\rangle_x \langle\, t \,\rangle_x = \langle t\, \tau \,\rangle_0 + \langle\, \tau^2 \,\rangle x - \langle\, \tau \,\rangle_0 (\langle\, t \,\rangle_0 + \langle\, \tau \,\rangle_0 x) \tag{300}$$

giving

$$\mathrm{Cov}_{t\tau|x} = \mathrm{Cov}_{t\tau|0} + x\, \sigma^2_\tau \tag{301}$$

The correlation coefficient is

$$\rho_{t\tau|x} = \frac{\mathrm{Cov}_{t\tau|x}}{\sigma_{t|x}\sigma_{\tau|x}} = \frac{\mathrm{Cov}_{t\tau|0} + x\sigma^2_\tau}{\sigma_{\tau|0}\sqrt{\sigma^2_{t|0} + 2\, x\, \mathrm{Cov}_{t\tau|0} + x^2\sigma^2_\tau}} \tag{302}$$

and we have that

$$\rho_{t\tau|x} \to 1 \qquad\qquad x \to \infty \tag{303}$$

Covariance Between Time and Frequency. It is interesting to calculate the covariance between time and frequency at a given position,

$$\mathrm{Cov}_{t\omega|x} = \langle t\,\omega\,\rangle_x - \langle t\,\rangle_x \langle\,\omega\,\rangle_x \tag{304}$$

Now

$$\begin{aligned}
\langle\, t\omega\,\rangle_x &= \tfrac{1}{2}\langle\,\omega\mathcal{T} + \mathcal{T}\omega\,\rangle_x &(305)\\
&= -\int \omega\,\frac{\partial\psi(\omega,x)}{\partial\omega}|F(\omega,x)|^2\,d\omega &(306)\\
&= -\int \omega\,[\,\frac{\partial\psi(\omega,0)}{\partial\omega} - \tau x]|F(\omega,x)|^2\,d\omega &(307)\\
&= -\int \omega\,[\,\frac{\partial\psi(\omega,0)}{\partial\omega} - \tau x]|F(\omega,0)|^2\,d\omega &(308)
\end{aligned}$$

Hence

$$\langle t\,\omega\,\rangle_x = \langle t\,\omega\,\rangle_0 + \langle\omega\,\tau\,\rangle_0 x \tag{309}$$

and therefore

$$\begin{aligned}
\mathrm{Cov}_{t\omega|x} &= \langle\omega\, t\,\rangle_x - \langle\omega\,\rangle_x\langle\, t\,\rangle_x &(310)\\
&= \langle t\,\omega\,\rangle_0 + \langle\omega\,\tau\,\rangle_0 x - \langle\omega\,\rangle_0(\langle\, t\,\rangle_0 + Tx) &(311)\\
&= \mathrm{Cov}_{t\omega|0} + x\,\mathrm{Cov}_{\omega\tau|0} &(312)
\end{aligned}$$

where as usual

$$\mathrm{Cov}_{\omega\tau|0} = \langle\omega\,\tau\,\rangle_0 - \langle\omega\,\rangle_0\langle\,\tau\,\rangle_0 \tag{313}$$

Relation between frequency transit time and group velocity. If one interprets the group velocity as the velocity that a certain frequency has and if one interprets the frequency transit time as the time it takes a frequency to travel a unit distance, then clearly there should be a relation between the two. Let us think in terms of the analogy with runners. Suppose the velocity of a runner is v and hence in a time T he travels

$$x = vT \tag{314}$$

Now his frequency transit time τ is the time it takes to travel a unit distance, that is given by

$$\tau = T/x = 1/v \tag{315}$$

and hence we expect the relation between frequency transit time and group velocity to be

$$\tau(\omega) = 1/v(K(\omega)). \tag{316}$$

That is indeed the case.

8.1 Instantaneous Frequency and Local Spatial Frequency

As before we define amplitude and phase by,

$$u(x,t) = A(x,t)e^{i\varphi(x,t)} \tag{317}$$

but because of the change in convention for this part we define instantaneous frequency as

$$\omega_i(x,t) = \frac{\partial}{\partial t}\varphi(x,t) \tag{318}$$

and spatial local frequency by

$$k_i(x,t) = -\frac{\partial}{\partial t}\varphi(x,t) \tag{319}$$

Asymptotic solution. We now give the asymptotic solution for case B. One sets

$$K'(\omega_s) = t/x \tag{320}$$

and hence

$$u_a(x,t) \sim F(\omega,0)\sqrt{\frac{1}{xK''(\omega)}}e^{i\omega t - iK(\omega)x - i\pi \mathrm{sgn}\, K''/4}|_{\omega=\omega_s} \tag{321}$$

and the amplitude and phase are given by

$$|u_a(x,t)| = |F(\omega,0)|\sqrt{\frac{1}{xK''(\omega)}} \tag{322}$$

$$\varphi_a(x,t) = \psi(\omega,0) - K(\omega)x + \omega t - \pi \mathrm{sgn}\, K''/4 \tag{323}$$

Instantaneous Frequency. To obtain the instantaneous frequency we have

$$\omega_i(x,t) = \frac{\partial \varphi_a(x,t)}{\partial t} \tag{324}$$

$$= \frac{\partial \omega}{\partial t}\left[\frac{\partial \psi}{\partial \omega} - x\frac{dK(\omega)}{d\omega} + t\right] + \omega \tag{325}$$

which gives

$$\omega_i(x,t) = \left[\frac{\partial \omega}{\partial t}\frac{\partial \psi}{\partial \omega}\right] + \omega \tag{326}$$

and since

$$K''(\omega)\frac{\partial \omega}{\partial t} = 1/x \tag{327}$$

giving

$$\frac{\partial \omega}{\partial t} = \frac{1}{xK''(\omega)} = \frac{K'(k)}{tK''(k)} \tag{328}$$

Hence,

$$\omega_i(x,t) = \frac{1}{xK''(\omega)}\frac{\partial\psi}{\partial\omega} + \omega \tag{329}$$

$$= \frac{K'(k)}{tK''(k)}\frac{\partial\psi}{\partial\omega} + \omega \tag{330}$$

Spatial local frequency. The spatial instantaneous frequency is

$$k_i(x,t) = -\frac{\partial}{\partial x}\varphi_a(x,t) \tag{331}$$

$$= -\left[\frac{d\psi}{d\omega_s}\frac{\partial\omega_s}{\partial x} + t\frac{\partial\omega_s}{\partial x} - K(\omega_s) - x\frac{dK(\omega_s)}{d\omega_s}\frac{\partial\omega_s}{\partial x}\right] \tag{332}$$

$$= -\left[\frac{d\psi}{d\omega_s} + t - x\frac{dK(\omega_s)}{d\omega_s}\right]\frac{\partial\omega_s}{\partial x} + K(\omega_s) \tag{333}$$

$$= -\left[\frac{d\psi}{d\omega_s} + t - x\frac{t}{x}\right]\frac{\partial\omega_s}{\partial x} + K(\omega_s) \tag{334}$$

or

$$k_i(x,t) = -\frac{d\psi}{d\omega_s}\frac{\partial\omega_s}{\partial x} + K(\omega_s) \tag{335}$$

But from Eq. (320) we now have that

$$K''(\omega_s)\frac{\partial\omega_s}{\partial x} = -t/x^2 \tag{336}$$

giving

$$\frac{\partial\omega_s}{\partial x} = -\frac{t}{x^2K''(\omega_s)} \tag{337}$$

$$\tag{338}$$

and hence

$$k_i(x,t) = -\frac{t}{x^2K''(\omega_s)}\frac{d\psi}{d\omega_s} + K(\omega_s) \tag{339}$$

8.2 Wigner Distribution (Time-Frequency)

The Wigner distribution now is the conventional time-frequency one. Again we will use the standard notation for the signal but now for the spectrum we have

$$u(x,t) = A(x,t)e^{i\varphi(x,t)} \tag{340}$$

$$F(\omega,x) = \frac{1}{\sqrt{2\pi}}\int u(x,t)\,e^{-i\omega t}\,dt \tag{341}$$

$$= B(\omega,x)e^{i\psi(\omega,x)} \tag{342}$$

The time-frequency Wigner distribution is then

$$W(t,\omega,x) = \frac{1}{2\pi}\int u^*(x,t-\tfrac{1}{2}\tau)\,u(x,t+\tfrac{1}{2}\tau)\,e^{-i\tau\omega}\,d\tau \tag{343}$$

$$=\frac{1}{2\pi}\int F^*(\omega+\tfrac{1}{2}\theta,x)\,F(\omega-\tfrac{1}{2}\theta,x)\,e^{-i\theta t}\,d\theta \tag{344}$$

The instantaneous frequency is the first conditional moment and is given by

$$\langle\omega\rangle_{t,x} = \int \omega W(t,\omega,x)d\omega \tag{345}$$

$$= \frac{\partial}{\partial t}\varphi(t,x) \tag{346}$$

and the group delay is

$$\langle t\rangle_{\omega,x} = B(\omega,x)\int tW(t,\omega,x)dt \tag{347}$$

$$= -\frac{\partial}{\partial\omega}\psi(\omega,x) \tag{348}$$

Also the covariance between time and frequency is

$$\langle\omega t\rangle_x = \int t\omega W(t,\omega,x)dtd\omega \tag{349}$$

$$= \int t\frac{\partial\varphi(t,x)}{\partial t}|u(t,x)|^2dt \tag{350}$$

$$= -\int \omega\frac{\partial\psi(\omega,x)}{\partial\omega}|F(\omega,x)\,|^2 \tag{351}$$

Now for pulse propagation we have

$$F(\omega,x) = F(\omega,0)\;e^{-iK(\omega)x} \tag{352}$$

$$= B(\omega,0)e^{i\psi(\omega,0)-iK(\omega)x} \tag{353}$$

and

$$B(\omega,x) = B(\omega,0) \tag{354}$$

$$\psi(\omega,x) = \psi(\omega,0) - K(\omega)x \tag{355}$$

which gives

$$\langle t\rangle_{\omega,x} = -\frac{\partial}{\partial\omega}\psi(\omega,x) \tag{356}$$

$$= -\frac{\partial}{\partial\omega}\psi(\omega,0) + K'(\omega)x \tag{357}$$

$$= -\frac{\partial}{\partial\omega}\psi(\omega,0) + \tau(\omega)x \tag{358}$$

At $x = 0$
$$\langle t \rangle_{\omega,0} = -\frac{\partial}{\partial\omega}\psi(\omega, 0) \tag{359}$$
and therefore
$$\langle t \rangle_{\omega,x} = \langle t \rangle_{\omega,0} + \tau(\omega)x \tag{360}$$
and also
$$\langle \omega t \rangle_x = -\int \omega \frac{\partial\psi(\omega, x)}{\partial\omega} |F(\omega, x)\,|^2 \tag{361}$$
$$= -\int \omega(\frac{\partial}{\partial\omega}\psi(\omega, 0) - \tau(\omega)x)|F(\omega, 0)\,|^2 \tag{362}$$
$$= \langle \omega t \rangle_0 + \langle \omega\tau(\omega) \rangle_0 x \tag{363}$$

which agree with the results of Part B.

Now consider the relation of the Wigner distribution at position x and how it is related at position zero. We write
$$W(t, \omega, x) = \frac{1}{2\pi}\int F^*(\omega + \tfrac{1}{2}\theta, 0)\, F(\omega - \theta/2, 0)\, e^{-i\theta t} e^{i[K(\omega+\theta/2)-K(\omega-\theta/2)]x}\, d\theta \tag{364}$$
and also
$$W(t, \omega, 0) = \frac{1}{2\pi}\int F^*(\omega + \tfrac{1}{2}\theta, 0)\, F(\omega - \tfrac{1}{2}\theta, 0)\, e^{-i\theta t}\, d\theta \tag{365}$$
which gives,
$$F^*(\omega + \tfrac{1}{2}\theta, 0)\, F(\omega - \tfrac{1}{2}\theta, 0) = \int W(t, \omega, 0)\, e^{i\theta t}\, dt \tag{366}$$
Inserting this into Eq. (364) we obtain
$$W(t, \omega, x) = \frac{1}{2\pi}\iint W(t', \omega, 0)\, e^{-i\theta(t'-t)} e^{i[k(\omega+\theta/2)-k(\omega-\theta/2)]x}\, d\theta\, dt' \tag{367}$$
which as before we write as
$$W(t, \omega, x) = \frac{1}{2\pi}\int W(t', \omega, 0) L(t' - t, x)\, dt' \tag{368}$$
with
$$L(t' - t, \omega, x) = \int \; e^{-i\theta(t'-t)} e^{i[K(\omega+\theta/2)-K(\omega-\theta/2)]x}\, d\theta \tag{369}$$
Now expand $[k(\omega + \theta/2) - k(\omega - \theta/2)]$ in a power series in θ to obtain
$$K(\omega + \theta/2) - K(\omega - \theta/2) = \sum_{n=0}^{\infty} \frac{K^{(2n+1)}(\omega)}{(2n + 1)!} \frac{\theta^{2n+1}}{2^n} \sim \tau(\omega)\theta + \frac{1}{24}\tau^{(2)}(\omega)\theta^3 \quad \cdots \tag{370}$$
where $\mathrm{K}^{(2n+1)}(\omega)$ is the $2n + 1$ derivative with respect to ω. Keeping only the first term
$$L(t' - t, x) \sim \int \; e^{-i\theta(t'-t)} e^{i\tau(\omega)x\theta}\, d\theta = \delta(t' - t + \tau(\omega)x) \tag{371}$$

Substituting this into Eq. (368) we have

$$W(t,\omega,x) \sim W(t-\tau(\omega)x,\omega,0) \tag{372}$$

As an example consider

$$u(0,t) = (\alpha/\pi)^{1/4}\, e^{-\alpha t^2/2+i\beta t^2/2+i\omega_0 t} \tag{373}$$

and the Wigner distribution is

$$W(t,\omega,0) \;=\; \frac{1}{\pi}\, e^{-\alpha t^2-(\omega-\beta t-\omega_0)^2/\alpha} \tag{374}$$

and therefore

$$W(t,\omega,x) \sim W(t-\tau(\omega)x,\omega,0) \tag{375}$$

$$= \;\frac{1}{\pi}\, e^{-\alpha(t-\tau(\omega)x)^2-(\omega-\omega_0-\beta(t-\tau(\omega)x))^2/\alpha} \tag{376}$$

If we further assume that

$$K(\omega) = D\omega + \gamma\omega^2/2 \tag{377}$$

$$\tau(\omega) = D + \gamma\omega \tag{378}$$

We have

$$W(t,\omega,x) \;\sim\; W(t-\tau(\omega)x,\omega,0) \tag{379}$$

$$= W(t-(D+\omega\gamma)x),\omega,0) \tag{380}$$

$$= \frac{1}{\pi}\exp\left[-\alpha(t-(D+\omega\gamma)x)^2-(\omega-\beta(t-D-\omega\gamma x-\omega_0)^2/\alpha\right] \tag{381}$$

9 Space-Time Signals and Distributions

9.1 Time-Frequency/Spatial-Spatial Frequency Representations

In the previous sections we considered a joint representation for time and frequency for a given position. We also considered joint space and spatial frequency distribution for a given time. However as of now there is no way to handle relations that involve any mixed variables, for example position and frequency. In this section we show how one can define for pulse propagation a four dimensional representation involving jointly the four quantities, time, frequency, space, and spatial frequency and for which the previous ones are special cases. We will develop the material in this section in the following way. First we will derive the equations keeping everything general and then specialize to the pulse propagation case.

9.2 General results

What we seek is [7]

$$W(x,k,t,\omega) = \text{ Joint distribution of time, frequency, space and spatial-frequency} \tag{382}$$

We use the four dimensional Wigner distribution defined by

$$W(x,k,t,\omega) = \left(\frac{1}{2\pi}\right)^2 \iint u^*(x - \frac{1}{2}\tau_x, t - \frac{1}{2}\tau)\, u(x + \frac{1}{2}\tau_x, t + \frac{1}{2}\tau)\, e^{-i\tau_x k - i\tau\omega}\, d\tau\, d\tau_x \tag{383}$$

Now, the four dimensional Wigner distribution may be written in a number of different ways all of which are useful depending on the calculation being done. First, we define the two dimensional spectrum of the signal, $G(k,\omega)$, and also list our previous definitions

$$u(x,t) = A(x,t)e^{i\varphi(x,t)} \tag{384}$$

$$S(k,t) = \frac{1}{\sqrt{2\pi}} \int u(x,t)\ e^{-ikx}\, dx = B(k,t)e^{i\psi(k,t)} \tag{385}$$

$$F(\omega,x) = \frac{1}{\sqrt{2\pi}} \int u(x,t)\ e^{-j\omega t}\, dt = B(\omega,x)e^{i\psi(\omega,x)} \tag{386}$$

$$G(k,\omega) = \frac{1}{2\pi} \iint u(x,t)\, e^{-j\omega t - jkx}\, dt\, dx = L(k,\omega)\, e^{i\psi(k,\omega)} \tag{387}$$

With these definitions we have

$$W(x,k,t,\omega) = \left(\frac{1}{2\pi}\right)^2 \iint G^*(k + \frac{1}{2}\theta_x, \omega + \frac{1}{2}\theta)\, G(k - \frac{1}{2}\theta_x, \omega - \frac{1}{2}\theta)\, e^{-j\theta_x x - j\theta t}\, d\theta\, d\theta_x \tag{388}$$

and

$$W(x,k,t,\omega) = \left(\frac{1}{2\pi}\right)^2 \iint S^*(k + \frac{1}{2}\theta_x, t - \frac{1}{2}\tau)\, S(k - \frac{1}{2}\theta_x, t - \frac{1}{2}\tau)\, e^{-j\theta_x x - j\tau\omega}\, d\theta\, d\tau \tag{389}$$

and also

$$W(x,k,t,\omega) = \left(\frac{1}{2\pi}\right)^2 \iint F^*(\omega + \frac{1}{2}\theta, x - \frac{1}{2}\tau_x) F(\omega - \frac{1}{2}\theta, x + \frac{1}{2}\tau_x)\, e^{-jt\theta - j\tau_x k}\, d\theta\, d\tau_x \tag{390}$$

The marginals of the distribution are derived and given in the Appendix.

Moments. There are of course numerous moments that can be developed however we will give only relevant moments and give them in the order which gives the most insight into pulse propagation. The calculations are given in the appendix. Consider first $\langle\, \omega\, \rangle_{x,k,t}$, the instantaneous frequency for a given spatial point and spatial frequency at a given time,

$$\langle\, \omega\, \rangle_{x,k,t} = \int \omega W(x,k,t,\omega)\, d\omega \tag{391}$$

$$= \frac{1}{2\pi}\frac{1}{2i}\int \left[s^*(x-\frac{1}{2}\tau_x,t)\,\frac{\partial s(x+\frac{1}{2}\tau_x,t)}{\partial t} - \frac{\partial s^*(x-\frac{1}{2}\tau_x,t)}{\partial t}\,s(x+\frac{1}{2}\tau_x,t)\right] e^{-j\tau_x k}\,d\tau_x \tag{392}$$

In the Appendix we give different expressions for this and other quantities. Now consider the instantaneous frequency irrespective of spatial frequency. It is obtained by further integrating Eq. (392) with respect to the spatial variable,

$$\langle\omega\rangle_{x,t} = \int \omega W(x,k,t,\omega)\,d\omega dk \tag{393}$$

$$= \int \langle\,\omega\,\rangle_{x,k,t} dk \tag{394}$$

and this evaluates to

$$\langle\omega\rangle_{x,t} = A^2(x,t)\,\frac{\partial}{\partial t}\phi(x,t) \tag{395}$$

which is the standard result for instantaneous frequency. Now consider $\langle k\rangle_{x,t,\omega}$. Because of the symmetry of our definitions one can readily write expressions for $\langle\,k\,\rangle_{x,\omega,t}$ and $\langle k\rangle_{x,t}$. *Position.* The average position is similarly given by

$$\langle\,x\,\rangle_{k,t,\omega} = \int x\,C(x,k,t,\omega)\,dx \tag{396}$$

$$= \frac{1}{2\pi}\frac{1}{2i}\int\left(S^*(k,t-\frac{1}{2}\tau)\,\frac{\partial S(k,t+\frac{1}{2}\tau)}{\partial k} - \frac{\partial S^*(k,t-\frac{1}{2}\tau)}{\partial k}S(k,t+\frac{1}{2}\tau)\right)e^{-j\tau\omega}\,d\tau \tag{397}$$

We now average further to obtain the average position for given wave number,

$$\langle\,x\,\rangle_{k,t} = \int x\,C(x,k,t,\omega)\,dx\,d\omega \tag{398}$$

$$= B^2(k,t)\,\frac{\partial}{\partial k}\psi(k,t) \tag{399}$$

We now further average to obtain

$$\langle\,x\,\rangle_{t} = \int B^2(k,t)\,\frac{\partial}{\partial k}\psi(k,t)\,dk \tag{400}$$

Covariance. We now consider the covariance. In the appendix we show

$$\langle k\omega\rangle_{x,t} = A^2(x,t)\frac{\partial\phi(x,t)}{\partial t}\frac{\partial\phi(x,t)}{\partial x} \tag{401}$$

Calculation tool and physical interpretation. One can think of $\frac{\partial}{\partial t}\varphi(x,t)$ and $\frac{\partial}{\partial x}\varphi(x,t)$ as the frequency and spatial frequency in the x,t representation. That is we associate

$$\omega \rightarrow \frac{\partial}{\partial t}\varphi(x,t) \tag{402}$$

$$k \rightarrow \frac{\partial}{\partial x}\varphi(x,t) \tag{403}$$

Using this idea, answers can generally be written down immediately without recourse to calculations. For example, suppose we want $\langle k\rangle_{x,t}$, then we can immediately write

$$\langle k\rangle_{x,t} = A^2(x,t)\frac{\partial}{\partial x}\varphi(x,t) \tag{404}$$

The same viewpoint can be taken in the spectral domain. In the spectral domain

$$x \rightarrow -\frac{\partial}{\partial k}\psi(k,\omega) \tag{405}$$

$$t \rightarrow -\frac{\partial}{\partial \omega}\psi(k,\omega) \tag{406}$$

Suppose, that we want to calculate $\langle xt\rangle_{k,\omega}$, we simply write

$$\langle xt\rangle_{k,\omega} = B^2(k,\omega)\frac{\partial\psi(k,\omega)}{\partial k}\frac{\partial\psi(k,\omega)}{\partial \omega} \tag{407}$$

Using this method we have the following results. They can be checked independently by direct calculation. The local covariance of frequency and spatial frequency is

$$\mathrm{Cov}_{x,t}(k\omega) = A^2(x,t)\left(\frac{\partial\varphi(x,t)}{\partial t}\frac{\partial\varphi(x,t)}{\partial x} - \frac{\partial\varphi(x,t)}{\partial t} - \frac{\partial\varphi(x,t)}{\partial x}\right) \tag{408}$$

and

$$\mathrm{Cov}(k\omega) = \iint A^2(x,t)\left(\frac{\partial\varphi(x,t)}{\partial t}\frac{\partial\varphi(x,t)}{\partial x} - \frac{\partial\varphi(x,t)}{\partial t} - \frac{\partial\varphi(x,t)}{\partial x}\right)\,dx\,dt \tag{409}$$

Also

$$\langle xt\rangle_{k,\omega} = B^2(k,\omega)\frac{\partial\psi(k,\omega)}{\partial \omega}\frac{\partial\psi(k,\omega)}{\partial k} \tag{410}$$

9.3 Application to Pulse Propagation

The above results are general. We now specialize to the pulse propagation case. Pulse propagation is imposed when we take

$$S(k,t) = S(k,0)\ e^{-iW(k)t} \tag{411}$$

$$= B(k,0)e^{i\psi(k,0)-iW(k)t} \tag{412}$$

or

$$B(k,t) = B(k,0) \tag{413}$$

$$\psi(k,t) = \psi(k,0) - W(k)t \tag{414}$$

Consider

$$\langle x\rangle_{k,t,\omega} = |B(k,0)|^2 \left[-\frac{\partial}{\partial k}\psi(k,0) + W'(k)t\right] \delta(\omega - W(k)) \tag{415}$$

$$= |B(k,0)|^2 \left[\langle x\rangle_{k,0} + W'(k)t\right] \delta(\omega - W(k)) \tag{416}$$

This is an interesting result and can be interpreted as follows. The average of the pulse at a certain frequency, spatial frequency, and time, varies linearly with time and with a velocity equal to $v(k)$; but the only frequencies allowed are those given by $\omega - W(k)$, that is those that satisfy the dispersion relation. Of course this is intuitively obvious but we have developed the mathematics to describe the situation. If we average over all frequencies, then

$$\langle x\rangle_{k,t} = \int \langle x\rangle_{k,t,\omega}\, d\omega = |B(k,0)|^2 \left[\langle x\rangle_{k,0} + v(k)t\right] \tag{417}$$

This is exactly the result obtained in previously using the two dimensional Wigner distribution. Also, further integration over spatial frequency gives

$$\langle x\rangle_t = \int \langle x\rangle_{k,t}\, d\omega = \int |B(k,0)|^2 \left[\langle x\rangle_{k,0} + v(k)t\right] d\omega \tag{418}$$

$$= \langle x\rangle_0 + \langle v(k)\rangle t \tag{419}$$

$$= \langle x\rangle_0 + Vt \tag{420}$$

which is Eq. (51). But here we have obtained it in a very direct manner. Now consider $\langle x\rangle_{t,\omega}$

$$\langle x\rangle_{t,\omega} = \int \langle x\rangle_{k,t,\omega}\, dk = \int |B(k,0)|^2 \left[\langle x\rangle_{k,0} + v(k)t\right] \delta(\omega - W(k))\, dk \tag{421}$$

To simplify this we assume that the solution to the equation $\omega - W(k) = 0$ is $k = K(\omega)$. Then

$$\langle x\rangle_{t,\omega} = \left(\frac{\frac{\partial}{\partial k}\psi(k,0)}{W'(k)} - t\right) |B(k,0)|^2{}_{k=g(\omega)} \tag{422}$$

and this can be similarly interpreted.

10 Examples

10.1 Spread, contraction time, etc. for: $u(x,0) = (\alpha/\pi)^{1/4}\, e^{-\alpha x^2/2 + i\beta x^2/2 + ik_0 x}$

In this example all quantities can be solved for exactly and hence offers a good case for verifying the results we have obtained. For the dispersion relation we take

$$W(k) = ck + \gamma k^2/2 \tag{423}$$

and the initial pulse is taken to be,

$$u(x,0) = (\alpha/\pi)^{1/4}\, e^{-\alpha x^2/2 + i\beta x^2/2 + ik_0 x} \tag{424}$$

We define

$$\eta = \alpha - i\beta \tag{425}$$

so that

$$u(x,0) = (\alpha/\pi)^{1/4}\, e^{-\eta x^2/2 + + ik_0 x} \tag{426}$$

and also, for convenience we define

$$\alpha' = \frac{\alpha}{(\alpha^2+\beta^2)} \tag{427}$$

$$\beta' = \frac{\beta}{(\alpha^2+\beta^2)} \tag{428}$$

The initial spectrum is calculated as follows,

$$S(k,0) = \frac{1}{\sqrt{2\pi}} \int u(x,0)\; e^{-ikx}\, dx \tag{429}$$

$$= \frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}} \exp\left[-\frac{(k-k_0)^2}{2\eta}\right] \tag{430}$$

$$= \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha - i\beta}} \exp\left[-\frac{\alpha(k-k_0)^2}{2(\alpha^2+\beta^2)} - i\frac{\beta(k-k_0)^2}{2(\alpha^2+\beta^2)}\right] \tag{431}$$

$$= \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha - i\beta}}\; e^{-\alpha'(k-k_0)^2/2 - i\beta'(k-k_0)^2/2} \tag{432}$$

and the time dependent spectrum is therefore

$$S(k,t) = S(k,0)\, e^{-iW(k)t} \tag{433}$$

$$= \frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}} \exp\left[-\frac{(k-k_0)^2}{2\eta} - i(ck + \gamma k^2/2)t\right] \tag{434}$$

$$= \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha - i\beta}} \exp\left[-\frac{\alpha(k-k_0)^2}{2(\alpha^2+\beta^2)} - i\frac{\beta(k-k_0)^2}{2(\alpha^2+\beta^2)} - i(ck + \gamma k^2/2)t\right] \tag{435}$$

At the initial time, $t = 0$, the mean and standard deviations of x and k are

$$\langle\, x \,\rangle_0 = 0 \tag{436}$$

$$\langle\, k \,\rangle_0 = k_0 \tag{437}$$

$$\sigma^2_{x|0} = \frac{1}{2\alpha} \tag{438}$$

$$\sigma^2_{k|0} = \frac{\alpha^2 + \beta^2}{2\alpha} \tag{439}$$

Also the covariance is

$$\mathrm{Cov}_{xv} = \frac{\gamma\beta}{2\alpha} \tag{440}$$

The moments are calculated very simply from the time dependent spectrum. The average group velocity and averaged square are

$$\langle\, v \,\rangle = c + \gamma k_0 \tag{441}$$

$$\langle\, v^2 \,\rangle = \gamma^2 \frac{\alpha^2 + \beta^2}{2\alpha} - (c + \gamma k_0)^2 \tag{442}$$

and therefore the standard deviation of group velocity is

$$\sigma_v^2 = \langle\, v_g^2 \,\rangle - \langle\, v \,\rangle^2 \tag{443}$$

$$= \gamma^2 \frac{\alpha^2 + \beta^2}{2\alpha} \tag{444}$$

Hence

$$\langle\, x \,\rangle_t = (c + \gamma k_0)\, t \tag{445}$$

$$\langle\, x^2 \,\rangle_t = \frac{1}{2\alpha} \left[(1 + \beta\, \gamma\, t)^2 + \gamma^2 \alpha^2\, t^2 \right] + (c + \gamma k_0)^{\,2} t^2 \tag{446}$$

and the spread is

$$\sigma^2_{x|t} = \sigma^2_{x|0} \left[1 + 2\beta\, \gamma\, t + \gamma^2 (\alpha^2 + \beta^2)\, t^2 \right] \tag{447}$$

$$= \sigma^2_{x|0} + 2t \frac{\gamma\beta}{2\alpha} + t^2 \gamma^2 \frac{\alpha^2 + \beta^2}{2\alpha} \tag{448}$$

$$= \frac{1}{2\alpha} \left[1 + 2\beta\, \gamma\, t + \gamma^2 (\alpha^2 + \beta^2)\, t^2 \right] \tag{449}$$

$$= \frac{1}{2\alpha} \left[(1 + \beta\, \gamma\, t)^2 + \gamma^2 \alpha^2\, t^2 \right] \tag{450}$$

It is also interesting to write $\sigma_{x|t}$ in terms of $\sigma_{x|0}$

$$\sigma_{x|t}^2 = \sigma_{x|0}^2 \left[1 + 2\beta\gamma t + \gamma^2 \left(\frac{1}{4\sigma_{x|0}^4} + \beta^2\right) t^2\right] \tag{451}$$

$$= \sigma_{x|0}^2 \left[1 + 2\beta\gamma t + \gamma^2 \frac{\sigma_{k|0}^2}{\sigma_{x|0}^2} t^2\right] \tag{452}$$

The reason this point is important is that it shows that the spread in time is not a linear function of the original spread. So for example if we make the initial spread very small the spread in time will relatively increase. This may be counterintuitive but it is correct and is a reflection of the uncertainty principle.

Contraction and Spread. The pulse will contract for times

$$t_C \leq -\frac{2\beta}{\gamma(\alpha^2 + \beta^2)} \qquad t_M = -\frac{\beta}{\gamma(\alpha^2 + \beta^2)} \tag{453}$$

This can only happen if either γ or β are less than zero, but not both. The minimum width is

$$\sigma_{x|t_M}^2 = \sigma_{x|0}^2 - \frac{\mathrm{Cov}_{xv|0}^2}{\sigma_v^2} = \frac{\alpha^2}{\alpha^2 + \beta^2} \sigma_{x|0}^2 \tag{454}$$

Note that β determines whether the pulse will contract and it has to be negative for contraction. Also note that the covariance is negative for negative β.

One can also obtain the value of β that will maximize the time of contraction. That will be the case when $\alpha = \beta$

$$t_C \leq -\frac{1}{\gamma\beta} \qquad t_M = -\frac{1}{2\gamma\beta} = \frac{1}{2} t_C \tag{455}$$

and

$$\sigma_{x|t_M}^2 = \frac{1}{2} \sigma_{x|t_0}^2 \tag{456}$$

Thus, the pulse cannot get any narrower than half of its original width.

10.1.1 Exact Solution

Our approach has allowed us to calculate the above quantities exactly and the calculations have all been done simply and using only the initial pulse and spectrum. That is the advantage of our method. It is now interesting to verify these with the exact solution for the pulse. To obtain the exact solution we calculate,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,t) e^{ikx} \tag{457}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}} \int \exp\left[-\frac{(k-k_0)^2}{2\eta} - i(ck + \gamma k^2/2)t + ikx\right] \tag{458}$$

This integral can be done but the answer turns out very complicated. We have found that by making a simple transformation first one gets a more tractable answer. In particular letting $k \to k + k_0$ we have

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}} \int \exp\left[-\frac{k^2}{2\eta} - i[c(k+k_0) + \gamma(k+k_0)^2/2]t + i(k+k_0)x\right] \tag{459}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}} \int \exp\left[-k^2(\frac{1}{2\eta} + i\gamma t/2) + ik(x - ct - \gamma k_0 t) - i(ck_0 + \gamma {k_0}^2/2)t + ik_0 x\right] \tag{460}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}} \int \exp\left[-\frac{k^2}{2\eta}(1 + i\eta\gamma t) + ik(x - ct - \gamma k_0 t) + ik_0(x - ct) - i\gamma {k_0}^2 t/2\right] \tag{461}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}} \int \exp\left[-\frac{k^2}{2\eta}(1 + i\eta\gamma t) + ik(x - ct - \gamma k_0 t) + ik_0(x - ct) - i\gamma {k_0}^2 t/2\right] \tag{462}$$

Carrying out the integration one obtains

$$u(x,t) = \frac{(\alpha/\pi)^{1/4}}{\sqrt{1 + i\gamma\eta t}} \exp\left[-\frac{\eta}{2}\frac{(x - ct - k_0\gamma t)^2}{1 + i\gamma\eta t} + ik_0(x - ct) - i\gamma k_0^2 t/2\right] \tag{463}$$

Using the fact that

$$1 + i\gamma\eta t = 1 + \gamma\beta t + i\gamma\alpha t \tag{464}$$

we have that

$$|1 + i\gamma\eta t|^2 = (1 + \beta\,\gamma\,t)^2 + \gamma^2\alpha^2\,t^2 \tag{465}$$

$$= 2\alpha\sigma_{x|t}^2 \tag{466}$$

Therefore

$$u(x,t) = \frac{(\alpha/\pi)^{1/4}}{\sqrt{1 + i\gamma\eta t}} \exp\left[-\frac{\eta}{2}\frac{(x - ct - k_0\gamma t)^2}{2\alpha\sigma_{x|t}^2}(1 + \gamma\beta t - i\gamma\alpha t) + ik_0(x - ct) - i\gamma k_0^2 t/2\right] \tag{467}$$

Also,

$$\eta(1 - i\gamma\eta t) = (\alpha - i\beta)(1 + \gamma\beta t - i\gamma\alpha t) \tag{468}$$

$$= \alpha - i\beta - i\gamma t(\alpha^2 + \beta^2) \tag{469}$$

and hence

$$u(x,t) = \frac{(\alpha/\pi)^{1/4}}{\sqrt{1 + i\gamma\eta t}} \exp\left[-\frac{(x - ct - k_0\gamma t)^2}{4\alpha\sigma_{x|t}^2}[\alpha - i\beta - i\gamma t(\alpha^2 + \beta^2)] + ik_0(x - ct) - i\gamma k_0^2 t/2\right] \tag{470}$$

This allows one to express the answer in terms of phase and amplitude

$$u_e(x,t) = \frac{1}{(2\pi\sigma_{x|t}^2)^{1/4}} \exp\left[-\frac{(x - \langle\, x\,\rangle_t)^2}{4\sigma_{x|t}^2}\right]$$
$$\exp\left[i\frac{(x - \langle\, x\,\rangle_t)^2\,\{\beta + (\alpha^2+\beta^2)\gamma t\}}{4\alpha\sigma_{x|t}^2} + ik_0(x-ct) - ik_0^2\,\gamma t/2 - i\delta\right] \tag{471}$$

where

$$\delta = \frac{1}{2}\arctan\frac{\alpha\gamma t}{1+\beta\gamma t} \tag{472}$$

The amplitude and phase are

$$|u_e(x,t)| = \frac{1}{(2\pi\sigma_{x|t}^2)^{1/4}} \exp\left[-\frac{(x - \langle\, x\,\rangle_t)^2}{4\sigma_{x|t}^2}\right] \tag{473}$$

$$\varphi_e = \frac{(x - \langle\, x\,\rangle_t)^2\,\{\beta + (\alpha^2+\beta^2)\gamma t\}}{4\alpha\sigma_{x|t}^2} + k_0(x-ct) - k_0^2\,\gamma t/2 - \delta \tag{474}$$

$$= \frac{\beta(x - \langle\, x\,\rangle_t)^2}{4\alpha\sigma_{x|t}^2} + \frac{(x - \langle\, x\,\rangle_t)^2\,(\alpha^2+\beta^2)\gamma t}{4\alpha\sigma_{x|t}^2} + k_0(x-ct) - k_0^2\,\gamma t/2 - \delta \tag{475}$$

$$= \frac{(x - \langle\, x\,\rangle_t)^2\,\{\beta + (\alpha^2+\beta^2)\gamma t\}}{4\alpha\sigma_{x|t}^2} + k_0(x - \langle\, x\,\rangle_t) + k_0^2\gamma t/2 - \delta \tag{476}$$

$$= \frac{1}{4\gamma}(x - \langle\, x\,\rangle_t)^2 \frac{d}{dt}\ln\sigma_{x|t}^2 + k_0(x - \langle\, x\,\rangle_t) + k_0^2\gamma t/2 - \delta \tag{477}$$

We also note that the above can be simplified in a different way if one uses

$$x - ct = x - \langle\, x\,\rangle_t + k_0\gamma t \tag{478}$$

$$(x-ct)^2 = (x - \langle\, x\,\rangle_t)^2 + 2(x - \langle\, x\,\rangle_t)k_0\gamma t + k_0^2\gamma^2 t^2 \tag{479}$$

10.1.2 Asymptotic Solution

It is also interesting to obtain the classical asymptotic approximation and to compare to the exact result. We have that

$$W'(k) = c + \gamma k \tag{480}$$

$$W''(k) = \gamma \tag{481}$$

Setting

$$W'(k) = c + \gamma k = x/t \tag{482}$$

we have that

$$k_s = \frac{x - ct}{\gamma t} \tag{483}$$

Now,

$$W(k) = ck + \gamma k^2/2 \tag{484}$$

$$= \frac{1}{2\gamma t}(x - ct)(x + ct) \tag{485}$$

and also now which gives that

$$k_s x - W(k_s)t = \frac{(x - ct)^2}{2\gamma t} \tag{486}$$

Also

$$k_s - k_0 = \frac{1}{\gamma t}(x - ct - k_0\gamma t) \tag{487}$$

$$= \frac{1}{\gamma t}(x - \langle\, x\,\rangle_t) \tag{488}$$

where

$$\langle\, x\,\rangle_t = (c + k_0\gamma)\, t \tag{489}$$

The asymptotic solution is

$$u_a(x,t) \sim S(k_s, 0)\sqrt{\frac{1}{t\gamma}}e^{-i\pi/4}\exp\left[i\frac{(x-ct)^2}{2\gamma t}\right] \tag{490}$$

But

$$S(k,0) = \frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}}\ \exp\left[-\frac{(x - ct - k_0\gamma t)^2}{2\eta\gamma^2 t^2}\right] \tag{491}$$

$$= \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha - i\beta}}\ \exp\left[-\frac{\alpha(x - ct - k_0\gamma t)^2}{2(\alpha^2+\beta^2)\gamma^2 t^2} - i\frac{\beta(x - ct - k_0\gamma t)^2}{2(\alpha^2+\beta^2)\gamma^2 t^2}\right] \tag{492}$$

and therefore

$$u_a(x,t) \sim \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha - i\beta}}\sqrt{\frac{1}{\gamma t}}e^{-i\pi/4}\exp\left[-\frac{\alpha(x - ct - k_0\gamma t)^2}{2(\alpha^2+\beta^2)\gamma^2 t^2} - i\frac{\beta(x - ct - k_0\gamma t)^2}{2(\alpha^2+\beta^2)\gamma^2 t^2} + i\frac{(x-ct)^2}{2\gamma t}\right] \tag{493}$$

$$|u_a(x,t)|^2 = \frac{1}{\gamma t}\ \frac{(\alpha/\pi)^{1/2}}{\sqrt{\alpha^2+\beta^2}}\ \exp\left[-\frac{\alpha(x - \langle\, x\,\rangle_t)^2}{(\alpha^2+\beta^2)\gamma^2 t^2}\right] \tag{494}$$

$$\varphi_a \sim -\beta'\frac{(x - \langle\, x\,\rangle_t)^2}{2\gamma^2 t^2} + \frac{(x-ct)^2}{2\gamma t} \tag{495}$$

$$= -\beta'\frac{(x - \langle\, x\,\rangle_t)^2}{2\gamma^2 t^2} + \frac{(x - \langle\, x\,\rangle_t)^2}{2\gamma t} + k_0(x - \langle\, x\,\rangle_t) + \frac{1}{2}k_0\gamma t \tag{496}$$

$$= \frac{1}{2}(x - \langle\, x\,\rangle_t)^2\left[\frac{1}{\gamma t} - \frac{\beta'}{\gamma^2 t^2}\right] + k_0(x - \langle\, x\,\rangle_t) + \frac{1}{2}k_0^2\gamma t \tag{497}$$

where we have used

$$x - ct = x - \langle\, x\,\rangle_t + k_0\gamma t \tag{498}$$

$$(x - ct)^2 = (x - \langle\, x\,\rangle_t)^2 + 2(x - \langle\, x\,\rangle_t)k_0\gamma t + k_0^2\gamma^2 t^2 \tag{499}$$

10.1.3 Comparison of Exact with Asymptotic solution

We now compare the asymptotic solution with the exact. In comparing the magnitude of the asymptotic solution with the exact we see that they are the same if we approximate the conditional standard deviation for large times by keeping only the quadratic terms

$$\sigma^2_{x|t\to\infty} = \frac{1}{2\alpha}[(\alpha^2 + \beta^2)\gamma^2 t^2] \tag{500}$$

Whether this is a general feature of the asymptote solution needs further investigation. That is, can the asymptotic solution be obtained in the following way:. write the parameters of the initial pulse in terms of the initial position and initial spread and then substitute for them the exact moments we have shown how to calculate. Is then the asymptotic solution the one that is obtained when only the quadratic term in the spread is kept?

10.2 Example: $u(x, 0) = \delta(x)$

Suppose we take an impulse at $x = 0$

$$u(x, 0) = \delta(x) \tag{501}$$

The initial spectrum is given by

$$S(k, 0) = \frac{1}{\sqrt{2\pi}} \tag{502}$$

We consider the case where the dispersion relation is given by

$$W(k) = ck + \gamma k^2/2 \tag{503}$$

and hence

$$S(k, t) = \frac{1}{\sqrt{2\pi}} e^{-i(ck+\gamma k^2/2)t} \tag{504}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,t) e^{ikx} dk \tag{505}$$
$$= \frac{1}{2\pi} \int e^{ik(x-ct)-i\gamma k^2 t/2} dk \tag{506}$$
$$= \frac{1}{2\pi} \sqrt{\frac{2\pi}{i\gamma t}} \exp\left[i \frac{(x-ct)^2}{2\gamma t} \right] \tag{507}$$
$$= \sqrt{\frac{1}{2\pi i \gamma t}} \exp\left[i \frac{(x-ct)^2}{2\gamma t} \right] \tag{508}$$

The exact phase is therefore

$$\varphi(x,t) = \frac{(x-ct)^2}{2\gamma t} \tag{509}$$

and the exact instantaneous frequency is

$$\omega_i(x,t) = -\frac{\partial}{\partial t} \varphi(x,t) \tag{510}$$
$$= c\frac{(x-ct)}{\gamma t} + \frac{(x-ct)^2}{2\gamma t^2} \tag{511}$$
$$= \frac{(x-x_0-ct)}{2\gamma t^2} [2ct + (x-ct)] \tag{512}$$
$$= \frac{(x-ct)(x+ct)}{2\gamma t^2} \tag{513}$$

and the exact spatial instantaneous frequency

$$k_i(x,t) = \frac{\partial}{\partial x} \varphi(x,t) \tag{514}$$
$$= \frac{(x-ct)}{\gamma t} \tag{515}$$

We now obtain the asymptotic solution. Using

$$k_s = \frac{x-ct}{\gamma t} \tag{516}$$
$$W(k_s) = \frac{1}{2\gamma t^2} (x-ct)(x+ct) \tag{517}$$
$$k_s x - W(k_s) t = \frac{(x-ct)^2}{2\gamma t} \tag{518}$$

we have

$$u_a(x,t) \sim S(k_s, 0) \sqrt{\frac{1}{tW''(k_s)}} e^{ik_s x - iW(k_s)t - i\pi \operatorname{sgn} K''/4} \tag{519}$$
$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{1}{\gamma t}} e^{ik_s(x-ct) - \gamma k_s^2 t/2 - i\pi\gamma/4} \tag{520}$$
$$= \sqrt{\frac{1}{i2\pi\gamma t}} \exp\left[i \frac{(x-ct)^2}{2\gamma t} \right] \tag{521}$$

But

$$\varphi_a(x,t) = \frac{(x-ct)^2}{2\gamma t} \tag{522}$$

The relation between exact and asymptotic is

$$\varphi_a(x,t) = \varphi(x,t) \tag{523}$$

The instantaneous frequency (asymptotic) is therefore

$$\omega_i(x,t) \quad (asymptotic) = \omega_i(x,t) \quad (exact) \tag{524}$$

Now let us use the equation from the text to derive the instantaneous frequency,

$$\omega_i(x,t) = \frac{x}{t^2 W''(k_s)} \frac{d\psi}{dk_s} + W(k_s) \tag{525}$$

We note that

$$\frac{d\psi}{dk_s} = 0 \tag{526}$$

and therefore

$$\omega_i(x,t) = W(k_s) \tag{527}$$

$$= \frac{1}{2\gamma t^2}(x-ct)(x+ct) \tag{528}$$

Also

$$k_i(x,t) = \frac{1}{tW''(k_s)} \frac{d\psi}{dk_s} + k_s \tag{529}$$

$$= k_s \tag{530}$$

$$= \frac{(x-ct)}{\gamma t} \tag{531}$$

Which agrees with the above

10.3 Example: $u(x,0) = \delta(x-x_0)$

The reason we consider this example, even though we have just done $\delta(x)$ will become apparent

$$u(x,0) = \delta(x-x_0) \tag{532}$$

The initial spectrum is given by

$$S(k,0) = \frac{1}{\sqrt{2\pi}} e^{-ikx_0} \tag{533}$$

and hence

$$S(k,t) = \frac{1}{\sqrt{2\pi}} e^{-ikx_0 - i(ck+\gamma k^2/2)t} \tag{534}$$

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int S(k,t) e^{ikx} dk && (535)\\
&= \frac{1}{2\pi} \int e^{ik(x-x_0-ct) - i\gamma k^2 t/2} dk && (536)\\
&= \frac{1}{2\pi} \sqrt{\frac{2\pi}{i\gamma t}} \exp\left[i \frac{(x-x_0-ct)^2}{2\gamma t} \right] && (537)\\
&= \sqrt{\frac{1}{2\pi i\gamma t}} \exp\left[i \frac{(x-x_0-ct)^2}{2\gamma t} \right] && (538)
\end{aligned}$$

Which of course could have been written down immediately since the solution is translation invariant. The exact phase is therefore

$$\varphi(x,t) = \frac{(x-x_0-ct)^2}{2\gamma t} \tag{539}$$

and we can write down the answers immediately by letting $x \to x - x_0$

$$\begin{aligned}
\omega_i(x,t) &= -\frac{\partial}{\partial t} \varphi(x,t) && (540)\\
&= \frac{(x-x_0-ct)(x-x_0+ct)}{2\gamma t^2} && (541)
\end{aligned}$$

and the exact spatial instantaneous frequency is

$$\begin{aligned}
k_i(x,t) &= \frac{\partial}{\partial x} \varphi(x,t) && (542)\\
&= \frac{(x-x_0-ct)}{\gamma t} && (543)
\end{aligned}$$

But now we obtain the asymptotic solution. Using

$$k_s = \frac{x-ct}{\gamma t} \tag{544}$$

$$W(k_s) = \frac{1}{2\gamma t^2}(x-ct)(x+ct) \tag{545}$$

$$k_s x - W(k_s)t = \frac{(x-ct)^2}{2\gamma t} \tag{546}$$

$$\begin{aligned}
u_a(x,t) &\sim S(k_s,0) \sqrt{\frac{1}{tW''(k_s)}} e^{ik_s x - iW(k_s)t - i\pi \operatorname{sgn} K''/4} && (547)\\
&= \frac{1}{\sqrt{2\pi}} e^{-ik_s x_0} \sqrt{\frac{1}{\gamma t}} e^{ik_s(x-ct) - \gamma k_s^2 t/2 - i\pi\gamma/4} && (548)\\
&= \sqrt{\frac{1}{i2\pi\gamma t}} \exp\left[i\frac{(x-ct)^2}{2\gamma t} - i\frac{x-ct}{\gamma t} x_0 \right] && (549)
\end{aligned}$$

But

$$\varphi_a(x,t) = \frac{(x-ct)^2}{2\gamma t} - \frac{x-ct}{\gamma t}x_0 \tag{550}$$

$$= \frac{(x-ct-x_0)^2 - {x_0}^2}{2\gamma t} \tag{551}$$

Thus we see that one can not get the asymptotic solution form the previous case by letting $x \to x - x_0$. The relation between exact and asymptotic is

$$\varphi_a(x,t) = \varphi(x,t) - \frac{{x_0}^2}{2\gamma t} \tag{552}$$

The instantaneous frequency (asymptotic) is therefore

$$\omega_i(x,t) \qquad (asymptotic) = \omega_i(x,t) \qquad (exact) - \frac{{x_0}^2}{2\gamma t^2} \tag{553}$$

$$k_i(x,t) \qquad (asymptotic) = k_i(x,t) \qquad (exact) \tag{554}$$

Now let us use

$$\omega_i(x,t) = \frac{x}{t^2 W''(k_s)}\frac{d\psi}{dk_s} + W(k_s) \tag{555}$$

to derive the instantaneous frequency. We note that

$$\frac{d\psi}{dk_s} = -x_0 \tag{556}$$

and therefore

$$\omega_i(x,t) = \frac{x}{t^2\gamma}(-x_0) + W(k_s) \qquad (asymptotic) \tag{557}$$

$$= \frac{1}{2\gamma t^2}(x-ct)(x+ct) - \frac{xx_0}{\gamma t^2} \tag{558}$$

But

$$(x-ct)(x+ct) = (x-x_0-ct)(x-x_0+ct) - x_0^2 + 2xx_0 \tag{559}$$

and therefore

$$\omega_i(x,t) = \frac{1}{2\gamma t^2}(x-ct)(x+ct) - \frac{xx_0}{\gamma t^2} \tag{560}$$

$$= \frac{1}{2\gamma t^2}(x-x_0-ct)(x-x_0+ct) - \frac{{x_0}^2}{2\gamma t^2} \tag{561}$$

which is the same as Eq. (552). Also,

$$k_i(x,t) = \frac{1}{tW''(k_s)}\frac{d\psi}{dk_s} + k_s = \tag{562}$$

$$= \frac{1}{\gamma t}(-x_0) + \frac{x - ct}{\gamma t} \tag{563}$$

$$= \frac{x - x_0 - ct}{\gamma t} \tag{564}$$

which agrees with the above.

10.4 Example: $u(x,0) = e^{ik_0x}$

Suppose at $x = 0$ we take

$$u(x,0) = \frac{1}{\sqrt{2\pi}} e^{ik_0x} \tag{565}$$

The initial spectrum is given by

$$S(k,0) = \delta(k - k_0) \tag{566}$$

We consider the case where the dispersion relation is given by

$$W(k) = ck + \gamma k^2/2 \tag{567}$$

The time dependent spectrum is

$$S(k,t) = \delta(k - k_0)e^{-i(ck+\gamma k^2/2)t} \tag{568}$$

and the exact answer is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,t)e^{ikx}dk \tag{569}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-i(ck_0+\gamma k_0^2/2)t+ik_0x} \tag{570}$$

The exact phase is therefore

$$\varphi(x,t) = -(ck_0 + \gamma k_0^2/2)t + k_0x \tag{571}$$

and the exact instantaneous frequency is

$$\omega_i(x,t) = -\frac{\partial}{\partial t}\varphi(x,t) \tag{572}$$

$$= ck_0 + \gamma k_0^2/2 \tag{573}$$

$$= W(k_0) \qquad (exact) \tag{574}$$

and the exact spatial instantaneous frequency is

$$k_i(x,t) = k_0 \qquad (exact) \tag{575}$$

We now obtain the asymptotic solution. From before we have that

$$k_s = \frac{x - ct}{\gamma t} \tag{576}$$

$$k_sx - W(k_s)t = \frac{(x - ct)^2}{2\gamma t} \tag{577}$$

and therefore

$$u_a(x,t) \sim S(k_s,0)\sqrt{\frac{1}{tW''(k_s)}}e^{ik_sx-iW(k_s)t-i\pi\operatorname{sgn}K''/4} \tag{578}$$

$$= \frac{1}{\sqrt{i}}\delta(k_s-k_0)\sqrt{\frac{1}{\gamma t}}\exp\left[i\frac{(x-ct)^2}{2\gamma t}\right] \tag{579}$$

$$= \frac{1}{\sqrt{i}}\delta(\frac{x-ct}{\gamma t}-k_0)\sqrt{\frac{1}{\gamma t}}\exp\left[i\frac{(x-ct)^2}{2\gamma t}\right] \tag{580}$$

$$= \frac{1}{\sqrt{i}}\delta(x-ct-\gamma tk_0)\sqrt{\gamma t}\exp\left[i\frac{(x-ct)^2}{2\gamma t}\right] \tag{581}$$

and therefore the asymptotic phase is

$$\varphi_a(x,t) = \frac{(x-ct)^2}{2\gamma t} \tag{582}$$

$$= k_0^2\gamma t/2 \tag{583}$$

The instantaneous frequency (asymptotic) is (note that we must differentiate Eq. (582) and not Eq. (583)

$$\omega_i(x,t) = c\,\frac{(x-ct)}{\gamma t} + \frac{(x-ct)^2}{2\gamma t^2} \qquad (asymptotic) \tag{584}$$

$$= c\frac{(x-ct)}{\gamma t} + \frac{(x-ct)^2}{2\gamma t^2} \tag{585}$$

$$= \frac{2(x-ct)ct+(x-ct)^2}{2\gamma t^2} \tag{586}$$

$$= \frac{(x-ct)(x+ct)}{2\gamma t^2} \tag{587}$$

$$= W(k_s) \tag{588}$$

$$= W(k_0) \tag{589}$$

and for this case we have that the exact equals the asymptotic. Also,

$$k_i(x,t) = c\,\frac{(x-ct)}{\gamma t} \qquad (asymptotic) \tag{590}$$

$$= c\,k_s \qquad (asymptotic) \tag{591}$$

$$= c\,k_s \tag{592}$$

Now we use

$$\omega_i(x,t) = \frac{x}{t^2W''(k_s)}\frac{d\psi}{dk_s} + W(k_s) \tag{593}$$

and using

$$\frac{d\psi}{dk_s} = 0 \tag{594}$$

we have

$$\omega_i(x,t) = W(k_s) \qquad (asymptotic) \tag{595}$$

$$= W(k_0) \tag{596}$$

Which is the same as Eq. (574).

Also

$$k_i(x,t) = \frac{1}{tW''(k_s)}\frac{d\psi}{dk_s} + k_s \qquad (asymptotic) \tag{597}$$

$$= k_s \tag{598}$$

$$= k_0 \tag{599}$$

10.5 Example: $u(x,0) \ = e^{i\beta x^2/2+ik_0x}$

At $t = 0$ we take

$$u(x,0) \ = e^{i\beta x^2/2+ik_0x} \tag{600}$$

The initial spectrum is given by

$$S(k,0) = \sqrt{\frac{i}{\beta}} \exp\left[-i\frac{(k-k_0)^2}{2\beta}\right] \tag{601}$$

We consider the case where the dispersion relation is given by

$$W(k) = ck + \gamma k^2/2 \tag{602}$$

and hence

$$S(k,t) = \sqrt{\frac{i}{\beta}} \exp\left[-i\frac{(k-k_0)^2}{2\beta} - iW(k)t\right] \tag{603}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,t) e^{ikx} \tag{604}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{-i\beta}} \int \exp\left[i\frac{(k-k_0)^2}{2\beta} - i(ck + \gamma k^2/2)t + ikx\right] \tag{605}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{-i\beta}} \int \exp\left[-i\frac{k^2}{2\beta} - i[c(k+k_0) + \gamma(k+k_0)^2/2]t + i(k+k_0)x\right] \tag{606}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{-i\beta}} \int \exp\left[-k^2(-\frac{1}{2i\beta} + i\gamma t/2) + ik(x - ct - \gamma k_0 t) - i(ck_0 + \gamma {k_0}^2/2)t + ik_0 x\right] \tag{607}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{-i\beta}} \int \exp\left[\frac{k^2}{2\beta}(1 + i\beta\gamma t) + ik(x - ct - \gamma k_0 t) + ik_0(x - ct) - i\gamma {k_0}^2 t/2\right] \tag{608}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{-i\beta}} \int \exp\left[-i\frac{k^2}{2\beta}(1 + \beta\gamma t) + ik(x - ct - \gamma k_0 t) + ik_0(x - ct) - i\gamma {k_0}^2 t/2\right] \tag{609}$$

$$= \frac{1}{\sqrt{1+\gamma\beta t}} \exp\left[i\frac{\beta}{2}\frac{(x - ct - k_0\gamma t)^2}{1+\gamma\beta t} + ik_0(x - ct) - i\gamma k_0^2 t/2\right] \tag{610}$$

The exact phase is therefore

$$\varphi(x,t) = \frac{\beta}{2}\frac{(x - ct - k_0\gamma t)^2}{1+\gamma\beta t} + k_0(x - ct) - \gamma k_0^2 t/2 \tag{611}$$

and the exact instantaneous frequency is

$$\omega_i(x,t) = -\frac{\partial}{\partial t}\varphi(x,t) \tag{612}$$
$$= (c+k_0\gamma)\frac{\beta(x-ct-k_0\gamma t)}{(1+\gamma\beta t)} + \gamma\frac{\beta^2(x-ct-k_0\gamma t)^2}{2(1+\gamma\beta t)^2} + k_0 c + \gamma k_0^2/2 \tag{613}$$
$$= \frac{\beta(x-ct-k_0\gamma t)}{1+\gamma\beta t}\left[(c+k_0\gamma) + \gamma\frac{\beta(x-ct-k_0\gamma t)}{2(1+\gamma\beta t)}\right] + k_0 c + \gamma k_0^2/2 \tag{614}$$

and the exact spatial instantaneous frequency is

$$k_i(x,t) = \frac{\beta(x-ct-k_0\gamma t)}{(1+\gamma\beta t)} + k_0 \tag{615}$$

Thus it still remains a chirp but with a different chirp rate.

We now obtain the asymptotic solution. We have that

$$k_s = \frac{x-ct}{\gamma t} \tag{616}$$

$$W(k_s) = \frac{(x-ct)(x+ct)}{2\gamma t^2} \tag{617}$$
$$k_s - k_0 = \frac{x-ct-k_0\gamma t}{\gamma t} \tag{618}$$
$$k_s x - W(k_s)t = \frac{(x-ct)^2}{2\gamma t} \tag{619}$$

and therefore

$$u_a(x,t) \sim S(k_s,0)\sqrt{\frac{1}{tW''(k_s)}}e^{ik_s x - iW(k_s)t - i\pi \operatorname{sgn} K''/4} \tag{620}$$
$$= \sqrt{\frac{i}{\beta}}\exp\left[-i\frac{(k-k_0)^2}{2\beta}\right]\sqrt{\frac{1}{\gamma t}}e^{ik_s(x-ct)-\gamma k_s^2 t/2 - i\pi/4} \tag{621}$$
$$= \sqrt{\frac{1}{\gamma\beta t}}\exp\left[-i\frac{(x-ct-\gamma t k_0)^2}{2\beta\gamma^2 t^2} + i\frac{(x-ct)^2}{2\gamma t}\right] \tag{622}$$

where we have used the fact that

$$e^{-i\pi/4} = \frac{1}{\sqrt{i}} \tag{623}$$

and therefore the asymptotic phase is

$$\varphi_a(x,t) = -\frac{(x-ct-\gamma t k_0)^2}{2\beta\gamma^2 t^2} + \frac{(x-ct)^2}{2\gamma t} \tag{624}$$

$$= \frac{\beta\gamma t(x-ct)^2 - (x-ct-\gamma t k_0)^2}{2\beta\gamma^2 t^2} = \tag{625}$$

$$= \frac{(x-ct)^2(\beta\gamma t - 1) + 2(x-ct)\gamma t k_0 - (\gamma t k_0)^2}{2\beta\gamma^2 t^2} \tag{626}$$

$$= -\frac{(x-ct-\gamma t k_0)^2}{2\beta\gamma^2 t^2} + \frac{(x-ct-\gamma t k_0)^2}{2\gamma t} + \frac{\gamma t {k_0}^2}{2} + (x-ct-\gamma t k_0)k_0 \tag{627}$$

$$\omega_i(x,t) = \frac{x}{t^2 W''(k_s)}\frac{d\psi}{dk_s} + W(k_s) \tag{628}$$

We note that

$$\frac{d\psi}{dk_s} = -\frac{(k-k_0)}{\beta} \tag{629}$$

$$= -\frac{x-ct-k_0\gamma t}{\beta\gamma t} \tag{630}$$

and therefore

$$\omega_i(x,t) = -\frac{x}{\gamma t^2}\frac{x-ct-k_0\gamma t}{\beta\gamma t} + \frac{(x-ct)(x+ct)}{2\gamma t^2} \qquad (asymptotic) \tag{631}$$

$$= -x\frac{x-ct-k_0\gamma t}{\beta\gamma^2 t^3} + \frac{(x-ct)(x+ct)}{2\gamma t^2} \tag{632}$$

Also,

$$k_i(x,t) = \frac{1}{tW''(k_s)}\frac{d\psi}{dk_s} + k_s \qquad (asymptotic) \tag{633}$$

$$= -\frac{x-ct-k_0\gamma t}{\beta\gamma^2 t^2} + \frac{x-ct}{\gamma t} \tag{634}$$

$$= -\frac{x-ct-k_0\gamma t}{\beta\gamma^2 t^2} + \frac{x-ct-k_0\gamma t}{\gamma t} + k_0 \tag{635}$$

10.6 Example: $u(0,t) = e^{i\omega_0 t}$

We consider the case where the dispersion relation is given by

$$K(\omega) = \gamma\omega^2/2 \tag{636}$$

and at $x = 0$

$$u(0,t) = e^{i\omega_0 t} \tag{637}$$

Its spectrum is

$$F(\omega, 0) = \sqrt{2\pi}\delta(\omega - \omega_0) \tag{638}$$

and hence,

$$F(\omega, x) = \sqrt{2\pi}\delta(\omega - \omega_0)e^{-i\gamma\omega^2 x/2} \tag{639}$$

and therefore

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int F(\omega, x)e^{i\omega t}d\omega \tag{640}$$

$$= e^{-i\gamma\omega_0^2 x/2 + j\omega_0 t} \tag{641}$$

and we see that the phase is given by

$$\varphi(x,t) = -\gamma\omega_0^2 x/2 + \omega_0 t \tag{642}$$

which gives

$$\omega_i(x,t) = \frac{\partial}{\partial t}\varphi(x,t) \tag{643}$$

$$= \omega_0 \tag{644}$$

and the spatial local frequency by

$$k_i(x,t) = -\frac{\partial}{\partial t}\varphi(x,t) \tag{645}$$

$$= \gamma\omega_0^2/2 \tag{646}$$

We now use the equations derived for the asymptotic answer. We have to first obtain ω_s

$$K'(\omega_s) = t/x = \gamma\omega_s \tag{647}$$

and hence

$$\omega_s = \frac{t}{\gamma x} \tag{648}$$

Further for our case we have that

$$\frac{\partial\psi}{\partial\omega} = 0 \tag{649}$$

Now

$$\omega_i(x,t) = \frac{1}{xK''(\omega)}\frac{\partial\psi}{\partial\omega} + \omega_s \tag{650}$$

$$= \frac{t}{\gamma x} \tag{651}$$

and also

$$k_i(x,t) = \frac{t}{x^2K''(\omega_s)}\frac{d\psi}{d\omega_s} + K(\omega_s) \tag{652}$$

$$= K(\omega_s) \tag{653}$$

$$= \gamma\omega_s^2/2 \tag{654}$$

$$= \frac{\gamma t}{2x} \tag{655}$$

and we see the answers do not agree. We now explore why this should be case. Consider the asymptotic signal itself

$$u_a(x,t) \sim F(\omega,0)\sqrt{\frac{1}{xK''(\omega)}}e^{i\omega t - iK(\omega)x - i\pi\mathrm{sgn}\,K''/4}|_{\omega=\omega_s} \tag{656}$$

Let us first work out that

$$\omega t - K(\omega)x = \frac{t^2}{2\gamma x} \tag{657}$$

and here the asymptotic solution is

$$u_a(x,t) \sim F(\omega,0)\sqrt{\frac{1}{xK''(\omega)}}e^{i\omega t - iK(\omega)x - i\pi\mathrm{sgn}\,K''/4}|_{\omega=\omega_s} \tag{658}$$

$$= \sqrt{2\pi}\delta(\frac{t}{\gamma x} - \omega_0)\sqrt{\frac{1}{x\gamma}}\exp\left[i\frac{t^2}{2\gamma x}\right] \tag{659}$$

Hence because of the delta function we now obtain

$$\omega_i(x,t) = \frac{t}{\gamma x} = \omega_0 \tag{660}$$

and also

$$k_i(x,t) = \frac{\gamma t}{2x} \tag{661}$$

$$= \frac{\gamma^2\omega_0}{2} \tag{662}$$

which still does not agree with the correct answer. However if we take the asymptotic signal and rewrite it as

$$u_a(x,t) \sim \sqrt{2\pi}\delta(\frac{t}{\gamma x} - \omega_0)\sqrt{\frac{1}{x\gamma}}\exp\left[i\frac{t^2}{2\gamma x}\right] \tag{663}$$

$$= \sqrt{2\pi}\delta(\frac{t}{\gamma x} - \omega_0)\sqrt{\frac{1}{x\gamma}}\exp\left[i\frac{t^2}{2\gamma x}\right] \tag{664}$$

then one would get the right answer if one differentiates the phase. What this example shows is that a further clarification is needed when one deals with delta functions. In particular for this case it is not clear which should be done first, the delta function substitution or the differentiations.

10.7 Example: $u(0,t) = \delta(t-t_0)$

Suppose we take an impulse at $x = 0$

$$u(0,t) = \delta(t-t_0) \tag{665}$$

The initial spectrum is given by

$$F(\omega,0) = \frac{1}{\sqrt{2\pi}} e^{-i\omega t_0} \tag{666}$$

and hence

$$F(\omega,x) = \frac{1}{\sqrt{2\pi}} e^{-i\gamma\omega^2 x/2 - i\omega t_0} \tag{667}$$

the exact answer is

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int F(\omega,x) e^{i\omega t} d\omega && (668) \\ &= \frac{1}{2\pi} \int e^{-i\gamma\omega^2 x/2 - i\omega(t_0 - t)} d\omega && (669) \\ &= \frac{1}{2\pi} \sqrt{\frac{2\pi}{i\gamma x}} \exp\left[i \frac{(t-t_0)^2}{2\gamma x} \right] && (670) \end{aligned}$$

and we see that the phase is given by

$$\varphi(x,t) = \frac{(t-t_0)^2}{2\gamma x} \tag{671}$$

which gives

$$\begin{aligned} \omega_i(x,t) &= \frac{\partial}{\partial t} \varphi(x,t) && (672) \\ &= \frac{t-t_0}{\gamma x} && (673) \end{aligned}$$

and the spatial local frequency by

$$\begin{aligned} k_i(x,t) &= -\frac{\partial}{\partial x} \varphi(x,t) && (674) \\ &= \frac{(t-t_0)^2}{2\gamma x^2} && (675) \end{aligned}$$

We now use the formulas derived in the text

$$K'(\omega_s) = t/x = \gamma \omega_s \tag{676}$$

and hence

$$\omega_s = \frac{t}{\gamma x} \tag{677}$$

$$K(\omega_s) = \gamma\omega_s^2/2 = \frac{t^2}{2\gamma x^2} \tag{678}$$

Now

$$\frac{\partial\psi(\omega, 0}{\partial\omega} = -t_0 \tag{679}$$

and hence

$$\omega_i(x,t) = \frac{1}{xK''(\omega)}\frac{\partial\psi}{\partial\omega} + \omega_s \tag{680}$$

$$= \frac{1}{x\gamma}(-t_0) + \frac{t}{\gamma x} \tag{681}$$

$$= \frac{t-t_0}{\gamma x} \qquad (asymptotic) \tag{682}$$

which is the same as the exact.

$$k_i(x,t) = \frac{t}{x^2K''(\omega_s)}\frac{d\psi}{d\omega_s} + K(\omega_s) \tag{683}$$

$$= \frac{t}{x^2\gamma}(-t_0) + \frac{t^2}{2\gamma x^2} \tag{684}$$

$$= \frac{t^2 - tt_0}{2\gamma x^2} \tag{685}$$

consider the asymptotic signal itself.

$$u_a(x,t) \sim F(\omega,0)\sqrt{\frac{1}{xK''(\omega)}}e^{i\omega t - iK(\omega)x - i\pi \mathrm{sgn}\, K''/4}|_{\omega=\omega_s} \tag{686}$$

Let us first work out that

$$\omega t - K(\omega)x = \frac{t^2}{2\gamma x} \tag{687}$$

and here the asymptotic solution is

$$u_a(x,t) \sim F(\omega,0)\sqrt{\frac{1}{xK''(\omega)}}e^{i\omega t - iK(\omega)x - i\pi \mathrm{sgn}\, K''/4}|_{\omega=\omega_s} \tag{688}$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{1}{x\gamma}}\exp[i\frac{t^2}{2\gamma x} - \frac{t}{\gamma x}t_0] \tag{689}$$

10.8 Example: $u(0,t) = (\alpha/\pi)^{1/4}\, e^{-\alpha t^2/2+j\omega_0 t}$

For the signal

$$u(0,t) \;=\; (\alpha/\pi)^{1/4}\, e^{-\alpha t^2/2+j\omega_0 t} \tag{690}$$

the initial special spectrum is

$$F(\omega) = \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha}}\;\exp\left[-\frac{(\omega-\omega_0)^2}{2\alpha}\right] \tag{691}$$

and the exact solution is

$$u(x,t) = \frac{(\alpha/\pi)^{1/4}}{\sqrt{2\alpha}}\sqrt{\frac{1}{\frac{1}{2}\alpha - i\gamma x}}\;\;\exp\left[-\frac{\omega_0^2}{2\alpha} + \frac{(it+\frac{\omega_0}{\alpha})}{4(\frac{1}{2\alpha}-i\gamma x)}\right] \tag{692}$$

The phase and amplitude are

$$\varphi(x,t) = \frac{-\gamma x t^2 - \omega_0 t/\alpha^2 + \omega_0\gamma x/\alpha^2}{4\left[(\frac{1}{2\alpha})^2+\gamma^2x^2\right]} + \frac{1}{2}\arctan\frac{2\gamma x}{\alpha} \tag{693}$$

$$= \frac{-\gamma x\alpha^2t^2 - \omega_0 t + \omega_0\gamma x}{1+4\alpha^2\gamma^2x^2} + \frac{1}{2}\arctan\frac{2\gamma x}{\alpha} \tag{694}$$

$$|u(x,t)| = \frac{(\alpha/\pi)^{1/4}}{\sqrt{2\alpha}}\left(\frac{1}{\alpha/4+\gamma^2x^2}\right)^2\;\exp\left[-\frac{1}{2}\alpha\left(\frac{t^2-4\omega_0\gamma x(t-\omega_0\gamma x)}{1+4\alpha^2\gamma^2x^2}\right)\right] \tag{695}$$

This gives

$$\omega_i(x,t) = \frac{\omega_0 + 2\alpha^2\gamma x t}{1+4\alpha^2\gamma^2x^2} \tag{696}$$

This is a chirp even though a pure sine wave is being generated at $x=0$. In fact, even for $\omega = 0$ we have a chirp.

10.9 Example: $u(0,t) \;=\; e^{i\beta t^2/2+j\omega_0 t}$

We take

$$u(0,t) \;=\; e^{i\beta t^2/2+i\omega_0 t} \tag{697}$$

which is a chirp. We have

$$F(\omega,0) = \sqrt{\frac{i}{\beta}}\;\exp\left[-i\frac{(\omega-\omega_0)^2}{2\beta}\right] \tag{698}$$

and hence,

$$F(\omega,x) = \sqrt{\frac{i}{\beta}}\;\exp\left[-i\frac{(\omega-\omega_0)^2}{2\beta} - i\gamma\omega^2 x/2\right] \tag{699}$$

and therefore

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}}\int F(\omega,x)e^{i\omega t}d\omega && (700)\\
&= \sqrt{\frac{i}{2\pi\beta}}\int\;\exp\left[-i\frac{(\omega-\omega_0)^2}{2\beta} - i\gamma\omega^2 x/2 + i\omega t\right]d\omega && (701)\\
&= \sqrt{\frac{i}{2\pi\beta}}\int\;\exp\left[-i\frac{\omega^2}{2\beta} - i\gamma(\omega+\omega_0)^2 x/2 + i(\omega+\omega_0)t\right]d\omega && (702)\\
&= \sqrt{\frac{i}{2\pi\beta}}\int\;\exp\left[-i\frac{\omega^2}{2\beta}(1+\beta\gamma x) + i\omega(t-\gamma\omega_0 x) - i\gamma\omega_0^2 x/2 + i\omega_0 t\right]d\omega && (703)\\
&= \sqrt{\frac{1}{1+\beta\gamma x}}\;\exp\left[i\frac{\beta(t-\gamma\omega_0 x)^2}{2(1+\beta\gamma x)} - i\gamma\omega_0^2 x/2 + i\omega_0 t\right] && (704)
\end{aligned}$$

and we see that the phase is given by

$$\varphi(x,t) = \frac{\beta(t-\gamma\omega_0 x)^2}{2(1+\beta\gamma x)} - \omega_0^2\gamma x/2 + \omega_0 t \tag{705}$$

which gives

$$\begin{aligned}
\omega_i(x,t) &= \frac{\partial}{\partial t}\varphi(x,t) && (706)\\
&= \beta\frac{t-\gamma\omega_0 x}{1+\beta\gamma x} + \omega_0 && (707)
\end{aligned}$$

and the spatial local frequency by

$$\begin{aligned}
k_i(x,t) &= -\frac{\partial}{\partial x}\varphi(x,t) && (708)\\
&= \gamma\omega_0\frac{\beta(t-\gamma\omega_0 x)}{(1+\beta\gamma x)} + \beta\gamma\frac{\beta(t-\gamma\omega_0 x)^2}{(1+\beta\gamma x)^2} - \gamma\omega_0^2/2 && (709)
\end{aligned}$$

We now compare to the asymptotic solution. We have

$$K'(\omega) = \gamma\omega = \frac{t}{x} \tag{710}$$

and that gives

$$\omega = \frac{t}{\gamma x} \tag{711}$$

Therefore,

$$F(\omega = \frac{t}{\gamma x}) = \frac{(\beta/\pi)^{1/4}}{\sqrt{\beta}} \ \exp\left[-\frac{(t-\gamma\omega_0 x)^2}{\gamma^2 x^2 \beta}\right] \tag{712}$$

and hence

$$u(x,t) = \sqrt{\frac{1}{\gamma\beta x}} \ \exp\left[-i\frac{(t-\gamma\omega_0 x)^2}{2\gamma^2 x^2\beta} + i\frac{t^2}{2\gamma x} - i\pi \mathrm{sgn}\gamma/4\right] \tag{713}$$

This gives an instantaneous frequency

$$\omega_i = \frac{t}{\gamma x} \tag{714}$$

10.10 Example: $u(0,t) = (\alpha/\pi)^{1/4} \, e^{-\alpha t^2/2 + i\beta t^2/2 + i\omega_0 t}$

This example is mathematically identical to the example previously considered and hence we do not give the details but just the results. We take

$$K(\omega) = D\omega + \gamma\omega^2/2 \tag{715}$$

and the initial pulse is taken to be,

$$u(0,t) = (\alpha/\pi)^{1/4} \, e^{-\alpha t^2/2 + i\beta t^2/2 + i\omega_0 t} \tag{716}$$

Defining

$$\eta = \alpha - i\beta \tag{717}$$

so that

$$u(0,t) = (\alpha/\pi)^{1/4} \, e^{-\eta t^2/2 + + i\omega_0 t} \tag{718}$$

and where as before

$$\alpha' = \frac{\alpha}{(\alpha^2 + \beta^2)} \tag{719}$$

$$\beta' = \frac{\beta}{(\alpha^2 + \beta^2)} \tag{720}$$

The initial spectrum is

$$
\begin{aligned}
F(\omega,0) &= \frac{1}{\sqrt{2\pi}} \int u(0,t)\; e^{-i\omega t}\, dt && (721)\\
&= \frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}}\; \exp\left[-\frac{(\omega-\omega_0)^2}{2\eta}\right] && (722)\\
&= \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha-i\beta}}\; \exp\left[-\frac{\alpha(\omega-\omega_0)^2}{2(\alpha^2+\beta^2)} - i\frac{\beta(\omega-\omega_0)^2}{2(\alpha^2+\beta^2)}\right] && (723)\\
&= \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha-i\beta}}\; e^{-\alpha'(\omega-\omega_0)^2/2 - i\beta'(\omega-\omega_0)^2/2} && (724)
\end{aligned}
$$

and further

$$
\begin{aligned}
F(\omega,x) &= F(\omega,0)\, e^{-iK(\omega)x} && (725)\\
&= \frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}}\; \exp\left[-\frac{(\omega-\omega_0)^2}{2\eta} - i(D\omega+\gamma\omega^2/2)x\right] && (726)\\
&= \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha-i\beta}}\; \exp\left[-\frac{\alpha(\omega-\omega_0)^2}{2(\alpha^2+\beta^2)} - i\frac{\beta(\omega-\omega_0)^2}{2(\alpha^2+\beta^2)} - i(D\omega+\gamma\omega^2/2)x\right] && (727)
\end{aligned}
$$

At the initial position, $x = 0$, the mean and standard deviations of t and ω are

$$
\begin{aligned}
\langle\, t\,\rangle_0 &= 0 && (728)\\
\langle\, \omega\,\rangle_0 &= \omega_0 && (729)\\
\sigma^2_{t|0} &= \frac{1}{2\alpha} && (730)\\
\sigma^2_{\omega|0} &= \frac{\alpha^2+\beta^2}{2\alpha} && (731)
\end{aligned}
$$

and also

$$
\mathrm{Cov}_{t\tau} = \frac{\gamma\beta}{2\alpha} \qquad (732)
$$

The average transit time and averaged square are

$$
\begin{aligned}
\langle\, \tau\,\rangle &= D + \gamma\omega_0 && (733)\\
\langle\, \tau^2\,\rangle &= \gamma^2\frac{\alpha^2+\beta^2}{2\alpha} - (D+\gamma\omega_0)^2 && (734)
\end{aligned}
$$

and therefore the standard deviation of transit time is

$$
\begin{aligned}
\sigma^2_\tau &= \langle\, \tau^2\,\rangle - \langle\, \tau\,\rangle^2 && (735)\\
&= \gamma^2\frac{\alpha^2+\beta^2}{2\alpha} && (736)
\end{aligned}
$$

Hence

$$\langle\, t\,\rangle_x = (D+\gamma\omega_0)\,x \tag{737}$$

$$\langle\, t^2\,\rangle_x = \frac{1}{2\alpha}\left[1+2\beta\,\gamma\,x+\gamma^2(\alpha^2+\beta^2)\,x^2\right]+(D+\gamma\omega_0)^2\,x^2 \tag{738}$$

and the spread is

$$\sigma^2_{t|x} = \sigma^2_{t|0}\left[1+2\beta\,\gamma\,x+\gamma^2(\alpha^2+\beta^2)\,x^2\right] \tag{739}$$

$$= \sigma^2_{t|0}+2x\frac{\gamma\beta}{2\alpha}+x^2\gamma^2\frac{\alpha^2+\beta^2}{2\alpha} \tag{740}$$

$$= \frac{1}{2\alpha}\left[1+2\beta\,\gamma\,x+\gamma^2(\alpha^2+\beta^2)\,x^2\right] \tag{741}$$

$$= \frac{1}{2\alpha}\left[(1+\beta\,\gamma\,x)^2+\gamma^2\alpha^2\,x^2\right] \tag{742}$$

Also,

$$\sigma^2_{t|x} = \sigma^2_{t|0}\left[1+2\beta\gamma x+\gamma^2\left(\frac{1}{4\sigma^4_{t|0}}+\beta^2\right)\,x^2\right] \tag{743}$$

$$= \sigma^2_{t|0}\left[1+2\beta\gamma x+\gamma^2\frac{\sigma^2_{\omega|0}}{\sigma^2_{t|0}}x^2\right] \tag{744}$$

Contraction and Spread. The pulse will contract for positions

$$x_C \le -\frac{2\beta}{\gamma(\alpha^2+\beta^2)} \qquad x_M = -\frac{\beta}{\gamma(\alpha^2+\beta^2)} \tag{745}$$

This can only happen if either γ or β are less than zero, but not both. The minimum width is

$$\sigma^2_{t|x_M} = \sigma^2_{t|0}-\frac{\mathrm{Cov}^2_{t\tau|0}}{\sigma^2_\tau} = \frac{\alpha^2}{\alpha^2+\beta^2}\sigma^2_{t|0} \tag{746}$$

One can also obtain the value of β that will maximize the time of contraction. That will be the case when $\alpha=\beta$

$$x_C \le -\frac{1}{\gamma\beta} \qquad x_M = -\frac{1}{2\gamma\beta} = \frac{1}{2}x_C \tag{747}$$

and

$$\sigma^2_{t|x_M} = \frac{1}{2}\sigma^2_{t|x_0} \tag{748}$$

Exact Solution. To obtain the exact solution we calculate,

$$u(t,x) = \frac{1}{\sqrt{2\pi}}\int F(\omega,x)e^{i\omega t} \tag{749}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}}\int \exp\left[-\frac{(\omega-\omega_0)^2}{2\eta}-i(D\omega+\gamma\omega^2/2)x+i\omega t\right] \tag{750}$$

and hence

$$u(t,x) = \frac{1}{\sqrt{2\pi}}\frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}}\int \exp\left[-\frac{\omega^2}{2\eta} - i[D(\omega+\omega_0)+\gamma(\omega+\omega_0)^2/2]x + i(\omega+\omega_0)t\right] \quad (751)$$

$$= \frac{1}{\sqrt{2\pi}}\frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}}\int \exp\left[-\omega^2(\frac{1}{2\eta}+i\gamma x/2) + i\omega(t-Dx-\gamma\omega_0 x) - i(D\omega_0+\gamma{\omega_0}^2/2)x + i\omega_0 t\right] \quad (752)$$

$$= \frac{1}{\sqrt{2\pi}}\frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}}\int \exp\left[-\frac{\omega^2}{2\eta}(1+i\eta\gamma x) + i\omega(t-Dx-\gamma\omega_0 x) + i\omega_0(t-Dx) - i\gamma{\omega_0}^2 x/2\right] \quad (753)$$

$$= \frac{1}{\sqrt{2\pi}}\frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}}\int \exp\left[-\frac{\omega^2}{2\eta}(1+i\eta\gamma x) + i\omega(t-Dx-\gamma\omega_0 x) + i\omega_0(t-Dx) - i\gamma{\omega_0}^2 x/2\right] \quad (754)$$

Carrying out the integration one obtains

$$u(t,x) = \frac{(\alpha/\pi)^{1/4}}{\sqrt{1+i\gamma\eta x}}\exp\left[-\frac{\eta}{2}\frac{(t-Dx-\omega_0\gamma x)^2}{1+i\gamma\eta x} + i\omega_0(t-Dx) - i\gamma\omega_0^2 x/2\right] \quad (755)$$

Using

$$1+i\gamma\eta x = 1+\gamma\beta x + i\gamma\alpha x \quad (756)$$

we have that

$$|1+i\gamma\eta x|^2 = (1+\beta\,\gamma\,x)^2 + \gamma^2\alpha^2\,x^2 \quad (757)$$

$$= 2\alpha\sigma_{t|x}^2 \quad (758)$$

Therefore

$$u(t,x) = \frac{(\alpha/\pi)^{1/4}}{\sqrt{1+i\gamma\eta x}}\exp\left[-\frac{\eta}{2}\frac{(t-Dx-\omega_0\gamma x)^2}{2\alpha\sigma_{t|x}^2}(1+\gamma\beta x - i\gamma\alpha x) + i\omega_0(t-Dx) - i\gamma\omega_0^2 x/2\right] \quad (759)$$

Also,

$$\eta(1-i\gamma\eta x) = (\alpha - i\beta)(1+\gamma\beta x - i\gamma\alpha x) \quad (760)$$

$$= \alpha - i\beta - i\gamma x(\alpha^2+\beta^2) \quad (761)$$

and hence

$$u(t,x) = \frac{(\alpha/\pi)^{1/4}}{\sqrt{1+i\gamma\eta x}}\exp\left[-\frac{(t-Dx-\omega_0\gamma x)^2}{4\alpha\sigma_{t|x}^2}[\alpha - i\beta - i\gamma x(\alpha^2+\beta^2)] + i\omega_0(t-Dx) - i\gamma\omega_0^2 x/2\right] \quad (762)$$

and also,

$$u_e(t,x) = \frac{1}{(2\pi\sigma_{t|x}^2)^{1/4}} \exp\left[-\frac{(t-\langle\, t\,\rangle_x)^2}{4\sigma_{t|x}^2}\right]$$
$$\exp\left[i\frac{(t-\langle\, t\,\rangle_x)^2\left\{\beta+(\alpha^2+\beta^2)\gamma x\right\}}{4\alpha\sigma_{t|x}^2} + i\omega_0(t-Dx) - i\omega_0^2\,\gamma x/2 - i\delta\right] \tag{763}$$

where

$$\delta = \frac{1}{2}\arctan\frac{\alpha\gamma x}{1+\beta\gamma x} \tag{764}$$

The amplitude and phase are

$$|u_e(t,x)| = \frac{1}{(2\pi\sigma_{t|x}^2)^{1/4}} \exp\left[-\frac{(t-\langle\, t\,\rangle_x)^2}{4\sigma_{t|x}^2}\right] \tag{765}$$

$$\varphi_e = \frac{(t-\langle\, t\,\rangle_x)^2\left\{\beta+(\alpha^2+\beta^2)\gamma x\right\}}{4\alpha\sigma_{t|x}^2} + \omega_0(t-Dx) - \omega_0^2\,\gamma x/2 - \delta \tag{766}$$

$$= \frac{\beta(t-\langle\, t\,\rangle_x)^2}{4\alpha\sigma_{t|x}^2} + \frac{(t-\langle\, t\,\rangle_x)^2\,(\alpha^2+\beta^2)\gamma x}{4\alpha\sigma_{t|x}^2} + \omega_0(t-Dx) - \omega_0^2\,\gamma x/2 - \delta \tag{767}$$

$$= \frac{(t-\langle\, t\,\rangle_x)^2\left\{\beta+(\alpha^2+\beta^2)\gamma x\right\}}{4\alpha\sigma_{t|x}^2} + \omega_0(t-\langle\, t\,\rangle_x) + \omega_0^2\gamma x/2 - \delta \tag{768}$$

$$= \frac{1}{4\gamma}(t-\langle\, t\,\rangle_x)^2\frac{d}{dx}\ln\sigma_{t|x}^2 + \omega_0(t-\langle\, t\,\rangle_x) + \omega_0^2\gamma x/2 - \delta \tag{769}$$

We also note that the above can be simplified in different ways if one uses

$$t - Dx = t - \langle\, t\,\rangle_x + \omega_0\gamma x \tag{770}$$

$$(t-Dx)^2 = (t-\langle\, t\,\rangle_x)^2 + 2(t-\langle\, t\,\rangle_x)\omega_0\gamma x + \omega_0^2\gamma^2 x^2 \tag{771}$$

Asymptotic Solution. We have that

$$W'(\omega) = D + \gamma\omega \tag{772}$$

$$W''(\omega) = \gamma \tag{773}$$

Setting

$$W'(\omega) = D + \gamma\omega = t/x \tag{774}$$

we obtain

$$\omega_s = \frac{t-Dx}{\gamma x} \tag{775}$$

Now,

$$K(\omega) = D\omega + \gamma\omega^2/2 \tag{776}$$
$$= \frac{1}{2\gamma x}(t - Dx)(t + Dx) \tag{777}$$

and also

$$\omega_s t - K(\omega_s)x = \frac{(t - Dx)^2}{2\gamma x} \tag{778}$$

In addition

$$\omega_s - \omega_0 = \frac{1}{\gamma x}(t - Dx - \omega_0\gamma x) \tag{779}$$
$$= \frac{1}{\gamma x}(t - \langle\, t\,\rangle_x) \tag{780}$$

where

$$\langle\, t\,\rangle_x = (D + \omega_0\gamma)\, x \tag{781}$$

The asymptotic solution is

$$u_a(t, x) \sim F(\omega_s, 0)\sqrt{\frac{1}{x\gamma}}e^{-i\pi/4} \exp\left[i\frac{(t - Dx)^2}{2\gamma x}\right] \tag{782}$$

But

$$F(\omega, 0) = \frac{(\alpha/\pi)^{1/4}}{\sqrt{\eta}} \exp\left[-\frac{(t - Dx - \omega_0\gamma x)^2}{2\eta\gamma^2 x^2}\right] \tag{783}$$
$$= \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha - i\beta}} \exp\left[-\frac{\alpha(t - Dx - \omega_0\gamma x)^2}{2(\alpha^2 + \beta^2)\gamma^2 x^2} - i\frac{\beta(t - Dx - \omega_0\gamma x)^2}{2(\alpha^2 + \beta^2)\gamma^2 x^2}\right] \tag{784}$$

and therefore

$$u_a(t, x) \sim \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha - i\beta}}\sqrt{\frac{1}{\gamma x}}e^{-i\pi/4} \exp\left[-\frac{\alpha(t - Dx - \omega_0\gamma x)^2}{2(\alpha^2 + \beta^2)\gamma^2 x^2} - i\frac{\beta(t - Dx - \omega_0\gamma x)^2}{2(\alpha^2 + \beta^2)\gamma^2 x^2} + i\frac{(t - Dx)^2}{2\gamma x}\right] \tag{785}$$
$$|u_a(t, x)|^2 = \frac{1}{\gamma x}\ \frac{(\alpha/\pi)^{1/2}}{\sqrt{\alpha^2 + \beta^2}}\ \exp\left[-\frac{\alpha(t - \langle\, t\,\rangle_x)^2}{(\alpha^2 + \beta^2)\gamma^2 x^2}\right] \tag{786}$$

$$\varphi_a \sim -\beta'\frac{(t - \langle\, t\,\rangle_x)^2}{2\gamma^2 x^2} + \frac{(t - Dx)^2}{2\gamma x} \tag{787}$$
$$= -\beta'\frac{(t - \langle\, t\,\rangle_x)^2}{2\gamma^2 x^2} + \frac{(t - \langle\, t\,\rangle_x)^2}{2\gamma x} + \omega_0(t - \langle\, t\,\rangle_x) + \frac{1}{2}\omega_0\gamma x \tag{788}$$
$$= \frac{1}{2}(t - \langle\, t\,\rangle_x)^2\left[\frac{1}{\gamma x} - \frac{\beta'}{\gamma^2 x^2}\right] + \omega_0(t - \langle\, t\,\rangle_x) + \frac{1}{2}\omega_0^2\gamma x \tag{789}$$

where we have used

$$t - Dx = t - \langle\, t\,\rangle_x + \omega_0 \gamma x \tag{790}$$

$$(t - Dx)^2 = (t - \langle\, t\,\rangle_x)^2 + 2(t - \langle\, t\,\rangle_x)\omega_0\gamma x + \omega_0^2\gamma^2 x^2 \tag{791}$$

11 Future Research

There are a number of directions that the above idea of pulse propagation should be extended to.

1. The case with damping should be investigated and the formalism developed here should be extended to handle that case.

2. Can these methods presented be generalized to equations with non-constant coefficients?

3. The Gabor procedure for a pulse needs investigation. That is a proper analysis of the concept of an analytic signal for a pulse has not been done.

4. Instantaneous frequency and local spatial frequency have been obtained for the asymptotic solution. One should try to improve on this.

5. Exact calculation of moments. We have shown that the moments can be calculated exactly and easily. Can these moments be used to construct a better approximation than the classical asymptotic approximation? That is construct an approximation to $|u(x,t)|^2$ and in particular

$$\text{Find an approximation to: } |u(x,t)|^2 \;\; \text{Given :} \tag{792}$$

$$u(x,0) \;\; \text{and} \tag{793}$$

$$\langle\, x^n\,\rangle_t \quad n = 1, N \tag{794}$$

There are many methods to construct densities from a given set of moments and it would be interesting to apply these methods. Note that in this formulation one would only approximate $|u(x,t)|^2$, that is the magnitude. But if we add to this the time moments, then it may be possible to also get the phase to a better approximation.

6. Can the methods and models developed here by applied to nonlinear wave equations?

7. The accuracy of the Wigner approximation scheme needs further investigation.

8. Can one obtain equations of motion for the amplitude and phase separately? We write

$$u(x,t) = R(x,t)e^{i\varphi(x,t)} \tag{795}$$

We have been able to show that

$$\frac{\partial R}{\partial t} = \frac{1}{R} \operatorname{Im} u^*(x,t) W(\mathcal{K}) u(x,t) \tag{796}$$

$$\frac{\partial \varphi}{\partial t} = -\frac{1}{\rho} \operatorname{Re} u^*(x,t) W(\mathcal{K}) u(x,t) \tag{797}$$

Can these equations be solved directly for phase and amplitude?

9. The Wigner distribution approach should be generalized to other distributions \cite{cohen66,rev,book

12 References

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A Appendix: Notation for Part A

k	Spatial Frequency (wave number)
ω	Frequency
$u(x,t) = A(x,t)e^{i\varphi(x,t)}$	Pulse at position x and time t
$A(x,t), \varphi(x,t)$	Amplitude and phase of pulse
$\omega = W(k)$	Dispersion relation
$v(k) = W'(k)$	Group velocity
$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,0)\ e^{ikx - iW(k)t}\, dk$	General solution for a pulse
$S(k,0) = \frac{1}{\sqrt{2\pi}} \int u(x,0)\ e^{-ikx}\, dx$	Spatial spectrum at time zero
$S(k,t) = S(k,0)\ e^{-iW(k)t}$	Spatial spectrum at time t
$= \frac{1}{\sqrt{2\pi}} \int u(x,t)\ e^{-ikx}\, dx$	Spatial spectrum at time t
$S(k,t) = B(k,t)e^{j\psi(k,t)}$	Spatial spectrum in terms of amplitude and phase
$B(k,t),\ \psi(k,t)$	Amplitude and phase of spatial spectrum
$\mathcal{X} = i\,\frac{\partial}{\partial k}$	position operator in k space
$\langle\, x^n \,\rangle_t = \int x^n\, \lvert u(x,t)\rvert^2\, dx$	Spatial moments of a pulse at time t
$V = \int v(k)\, \lvert S(k,0)\rvert^2\, dk$	Average group velocity
σ_v^2	Standard deviation of group velocity
$\mathrm{Cov}_{xv\mid t}$	Covariance of position and group velocity at time t
$\rho_{xv\mid t}$	Correlation coefficient
t_C	Time of contraction
t_M	Time at minimum contraction
$\omega_i(x,t) = -\frac{\partial}{\partial t}\varphi(x,t)$	Instantaneous frequency
$k_i(x,t) = \frac{\partial}{\partial x}\varphi(x,t)$	Local spatial frequency

B Appendix: Notation for Part B

k	Spatial Frequency (wave number)
ω	Frequency
$u(x,t) = A(x,t)e^{i\varphi(x,t)}$	Pulse at position x and time t
$A(x,t), \varphi(x,t)$	Amplitude and phase of pulse
$k = K(\omega)$	Dispersion relation
$\tau(\omega) = K'(\omega)$	Transit time
$u(x,t) = \frac{1}{\sqrt{2\pi}} \int F(\omega,0)\, e^{i\omega t - iK(\omega)x}\, d\omega$	General solution for a pulse
$F(k,0) = \frac{1}{\sqrt{2\pi}} \int u(x,0)\, e^{-i\omega t}\, dt$	Spectrum at time zero
$F(\omega,x) = F(k,0)\, e^{-iK(\omega)x}$	Sspectrum at position x
$F(\omega,x) = \frac{1}{\sqrt{2\pi}} \int u(x,t)\, e^{-i\omega t}\, dt$	Spectrum at position x
$F(\omega,x) = B(\omega,x)e^{j\psi(\omega,x)}$	Spectrum in terms of amplitude and phase
$B(\omega,x),\ \psi(\omega,x)$	Amplitude and phase of spectrum
$\mathcal{T} = i\,\frac{\partial}{\partial\omega}$	Time operator in ω space
$\langle\, t^n \,\rangle_x = \int t^n\, \lvert u(x,t)\rvert^2\, dt$	Time Moments of a pulse at x
$T = \int \tau(\omega)\, \lvert F(k,0)\rvert^2\, d\omega$	Mean transit time
σ_τ^2	Standard deviation of transit time
$\mathrm{Cov}_{t\tau\vert x}$	Covariance of time and transit time at x
$\rho_{t\tau\vert x}$	Correlation coefficient
x_C	Positions where pulse is contracted
x_M	Position of minimum contraction
$\omega_i(x,t) = \frac{\partial}{\partial t}\varphi(x,t)$	Instantaneous (time) frequency
$k_i(x,t) = -\frac{\partial}{\partial x}\varphi(x,t)$	Local spatial frequency

C Appendix: Moments

In this appendix we give the derivations of the moments of a pulse. As pointed out in the text the spatial moments of a pulse as a function of time can be calculated directly and easily from the initial spectrum. In particular we have that in general:

$$\langle x^n \rangle_t = \int x^n \, |u(x,t)|^2 \, dx \tag{798}$$

$$= \int S^*(k,t) \, \mathcal{X}^n \, S(k,t) \, dk \tag{799}$$

where $\mathcal{X}$ is the position operator in the k representation

$$\mathcal{X} = i \, \frac{\partial}{\partial k} \tag{800}$$

And again as we pointed out, what makes the calculation easy in the spectral Fourier domain is that the time dependent spectrum is

$$S(k,t) = S(k,0) \, e^{-iW(k)t} \tag{801}$$

First we list here different expressions for the moments that are useful

$$\langle x \rangle_t = \int x \, |u(x,t)|^2 \, dx \tag{802}$$

$$= \int S^*(k,t) \mathcal{X} S(k,t) \, dk \tag{803}$$

$$\langle x^2 \rangle_t = \int x^2 \, |u(x,t)|^2 \, dx \tag{804}$$

$$= \int S^*(k,t) \mathcal{X}^2 S(k,t) \, dk \tag{805}$$

$$= \int \, |\mathcal{X} S(k,t)|^2 \, dk \tag{806}$$

$$\sigma_{x|t} = \int (x - \langle x \rangle_t)^2 \, |u(x,t)|^2 \, dx \tag{807}$$

$$= \langle x^2 \rangle_t - (\langle x \rangle_t)^2 \tag{808}$$

$$= \int S^*(k,t) (\mathcal{X} - \langle x \rangle_t)^2 S(k,t) \, dk \tag{809}$$

$$= \int |(\mathcal{X} - \langle x \rangle_t) S(k,t)|^2 \, dk \tag{810}$$

First Moment. We have

$$\langle x \rangle_t = \int x \, |u(x,t)|^2 \, dx = \int S^*(k,t) \, \mathcal{X} \, S(k,t) \, dk \tag{811}$$

but

$$\mathcal{X}S(k,t) = i\frac{\partial}{\partial k}S(k,t) = \left(i\frac{\partial}{\partial k}S(k,0) + tvS(k,0)\right)e^{-iW(k)t} \tag{812}$$

and therefore

$$\langle x\rangle_t = \int S^*(k,0)\mathcal{X}S(k,0)\,dk + t\int v|S(k,0)|^2\,dk \tag{813}$$

$$= \langle x\rangle_0 + t\int v|S(k,0)|^2\,dk \tag{814}$$

$$= \langle x\rangle_0 + t\langle v\rangle_0 \tag{815}$$

Second Moment. For the second moment we do it two different ways. First by way of Eq. (806) and then directly by way of Eq. (805). We calculate

$$|\mathcal{X}S(k,t)|^2 = \left|\left(i\frac{\partial}{\partial k}S(k,0) + tvS(k,0)\right)e^{-iW(k)t}\right|^2 \tag{816}$$

$$= \left|\left(i\frac{\partial}{\partial k}S(k,0) + tvS(k,0)\right)\right| \tag{817}$$

$$= \left(-i\frac{\partial}{\partial k}S^*(k,0) + tvS^*(k,0)\right)\left(i\frac{\partial}{\partial k}S(k,0) + tvS(k,0)\right) \tag{818}$$

$$= |\mathcal{X}S(k,0)|^2 + t^2\,v^2\,|S(k,0)|^2 + i\,t[vS^*(k,0)\frac{\partial}{\partial k}S(k,0) - vS(k,0)\frac{\partial}{\partial k}S_k^*(k,0)] \tag{819}$$

and therefore

$$\langle\, x^2\,\rangle_t = \int |\mathcal{X}S(k,0)|^2\,dk + t^2\int v^2\,|S(k,0)|^2\,dk$$

$$= i\,t\int\left[[vS^*(k,0)\frac{\partial}{\partial k}S(k,0) - vS(k,0)\frac{\partial}{\partial k}S_k^*(k,0)\right]dk \tag{820}$$

$$= i\,t\int\left[vS^*(k,0)\frac{\partial}{\partial k}S(k,0) + S^*(k,0)\frac{\partial}{\partial k}vS(k,0)\right] \tag{821}$$

$$= \langle\, x^2\,\rangle_0 + t\int S^*(k,0)[v,\mathcal{X}]_+S(k,0)\,dk + t^2\,\langle\, v^2\,\rangle \tag{822}$$

$$= \langle\, x^2\,\rangle_0 + t\,\langle\,[v,\mathcal{X}]_+\,\rangle_0 + t^2\,\langle\, v^2\,\rangle \tag{823}$$

where

$$[v,\mathcal{X}]_+ = iv\frac{\partial}{\partial k} + i\frac{\partial}{\partial k}v \tag{824}$$

$$= v\mathcal{X} + \mathcal{X}v \tag{825}$$

We now do it the second way, that is by way of Eq. (805),

$$\mathcal{X}^2 S(k,t) = i^2 \frac{\partial^2}{\partial k^2} S(k,t) \tag{826}$$

$$= i^2 \frac{\partial^2}{\partial k^2} S(k,0)\, e^{-iW(k)t} \tag{827}$$

$$= \left[-\frac{\partial^2}{\partial k^2} S(k,0) + it \left(\frac{dv}{dk} S(k,0) + 2v \frac{\partial S}{\partial k} \right) + v^2 t^2 S(k,0) \right] e^{-iW(k)t} \tag{828}$$

and therefore we obtain

$$\langle x^2 \rangle_t = \langle x^2 \rangle_0 + it \int v \left(S^*(k,0) \frac{d}{d} S(k,0) k - S(k,0) \frac{dS^*(k,0)}{dk} \right) dk + t^2 \langle v^2 \rangle_0 \tag{829}$$

$$= \langle x^2 \rangle_0 + it \int S^*(k,0) \left(v \frac{d}{dk} + \frac{d}{dk} v \right) S(k,0)\, dk + t^2 \langle v^2 \rangle_0 \tag{830}$$

$$= \langle\, x^2 \,\rangle_0 + t \,\langle\, [v, \mathcal{X}]_+ \,\rangle_0 + t^2 \,\langle\, v^2 \,\rangle \tag{831}$$

as before.

Standard Deviation or Spread. The standard deviation is therefore

$$\sigma^2_{x|t} = \langle\, x^2 \,\rangle_t - \langle\, x \,\rangle_t^2 = \langle\, x^2 \,\rangle_0 + t^2 \,\langle\, v^2 \,\rangle + t \,\langle\, [v, \mathcal{X}]_+ \,\rangle_0 - [\,\langle\, x \,\rangle + t \langle\, v \,\rangle\,]^2 \tag{832}$$

or

$$\sigma^2_{x|t} = \sigma^2_{x|0} + 2t\, \mathrm{Cov}_{xv} + t^2\, \sigma^2_v \tag{833}$$

where

$$\mathrm{Cov}(vx) = \frac{1}{2} \langle v\mathcal{X} + \mathcal{X} v \rangle_0 - \langle v \rangle_0 \langle x \rangle_0 \tag{834}$$

Global k moments. We have

$$\langle\, k \,\rangle_t = \int k\, |S(k,t)|^2\, dk \tag{835}$$

$$= \int u^*(x,t) \mathcal{K} u(x,t)\, dx \tag{836}$$

$$\langle\, k^2 \,\rangle_t = \int k^2\, |S(k,t)|^2\, dk \tag{837}$$

$$= \int u^*(x,t) \mathcal{K}^2 u(x,t)\, dx \tag{838}$$

$$= \int |\mathcal{K} u(x,t)|^2\, dx \tag{839}$$

$$\sigma_{k|t} = \int (k - \langle\, k \,\rangle_t)^2 |S(k,t)|^2\, dk \tag{840}$$

$$= \langle\, k^2 \,\rangle_t - (\langle\, k \,\rangle_t)^2 \tag{841}$$

$$= \int u^*(x,t) (\mathcal{K} - \langle\, k \,\rangle_t)^2 u(x,t)\, dx \tag{842}$$

$$= \int |(\mathcal{K} - \langle\, k \,\rangle_t)^2) u(x,t)|^2\, dx \tag{843}$$

In the case of wave propagation these moments are constant in time.

Covariance. To calculate the time dependence of the covariance between position and wave number we first note some general properties of the anticommutator operator when the variables are x, k. Using the commutator relation

$$[\mathcal{X}, \mathcal{K}] = i \tag{844}$$

we have that

$$[\mathcal{X}, \mathcal{K}]_+ = 2\mathcal{K}\mathcal{X} + i = 2\mathcal{X}\mathcal{K} - i \tag{845}$$

Also for two arbitrary functions a and b

$$[\mathcal{X}, \mathcal{K}]_+ ab = a[\mathcal{X}, \mathcal{K}]_+ b + b[\mathcal{X}, \mathcal{K}]_+ a - ab = 2k\mathcal{X}ab + iab \tag{846}$$

Now applying this to $S(k,0)\, e^{-iW(k)t}$ we have that

$$\tag{847}$$

$$[\mathcal{X}, \mathcal{K}]_+ S(k,0)\, e^{-iW(k)t} = 2ik(S_k(k,0) - iv\, t\, S(k,0))\, e^{-iW(k)t} + S(k,0)\, e^{-iW(k)t} \tag{848}$$

$$= e^{-iW(k)t}[\mathcal{X}, \mathcal{K}]_+ S(k,0) + 2kv\, tS(k,0)\, e^{-iW(k)t} \tag{849}$$

Hence

$$\langle\, xk\,\rangle_t = \langle\, \frac{1}{2}[\mathcal{X}, \mathcal{K}]_+\,\rangle = \int \frac{1}{2} S^*(k,0)[\mathcal{X}, \mathcal{K}]S(k,0)\, dk + t\int kv\, |S(k,0))|^2\, dk \tag{850}$$

or

$$\langle\, xk\,\rangle_t = \langle\, xk\,\rangle_0 + t\langle\, xv\,\rangle_0 \tag{851}$$

Thus $\langle\, xk\,\rangle_t$ increases linearly. Using the fact that all wave number averages are independent of time we have

$$\mathrm{Cov}_{xk}(t) = \langle\, xk\,\rangle_t - \langle\, x\,\rangle_t\langle\, k\,\rangle_t \tag{852}$$

$$= \langle\, xk\,\rangle_t - \langle\, x\,\rangle_t\langle\, k\,\rangle_0 \tag{853}$$

$$= \langle\, xk\,\rangle_0 + t\,\langle\, xv\,\rangle_0 - \langle\, k\,\rangle_0(\langle\, x\,\rangle_0 + t\langle\, v\,\rangle) \tag{854}$$

or

$$\mathrm{Cov}_{xk}(t) = \mathrm{Cov}_{xk}(0) + t\,\mathrm{Cov}_{xv}(0) \tag{855}$$

The Covariance

We now consider the covariance between x and k,

$$\mathrm{Cov}_{xk|t} = \langle x\, k\,\rangle_t - \langle x\,\rangle_0\langle\, k\,\rangle_0 \tag{856}$$

We have

$$\langle\, xk \,\rangle_t = \tfrac{1}{2}\langle\, k\mathcal{X} + \mathcal{X}k \,\rangle_t = -\int k\,\frac{\partial\psi(k,t)}{\partial k}|S(k,t)|^2\,dk \tag{857}$$

and

$$\langle x\,k\,\rangle_t = \langle x\,k\,\rangle_0 + \langle k\,v\,\rangle t \tag{858}$$

Inserting this into Eq. (855) gives

$$\mathrm{Cov}_{xk|t} = \mathrm{Cov}_{xk|0} + t\,\mathrm{Cov}_{kv} \tag{859}$$

where

$$\mathrm{Cov}_{kv} = \langle k\,v\,\rangle - \langle k\,\rangle\langle\, v\,\rangle \tag{860}$$

We note that Cov_{kv} is independent of time.

Also, one can obtain that

$$\frac{1}{2}\langle\, v\mathcal{X} + \mathcal{X}v\,\rangle_t = \frac{1}{2}\langle\, v\mathcal{X} + \mathcal{X}v\,\rangle_0 + \langle\, v^2(k)\,\rangle t \tag{861}$$

and hence

$$\mathrm{Cov}_{xk||t} = \mathrm{Cov}_{xk|0} + \sigma_v^2 t \tag{862}$$

$$\rho_{xk|t} = \frac{\mathrm{Cov}_{AK|t}}{\sigma_{x|t}\sigma_{k|t}} = \frac{\mathrm{Cov}_{xk}(t) = \mathrm{Cov}_{xk}(0) + \sigma_v^2 t}{\sigma_v\sqrt{\sigma_{x|0}^2 + 2\,t\,\mathrm{Cov}_{xv} + t^2\sigma_v^2}} \tag{863}$$

As before we have that

$$\rho_{xv|t} \to 1 \qquad \text{as} \qquad t \to \infty \tag{864}$$

D Appendix: Marginals of Space-Time Distributions

We give the marginals of the four dimensional distribution given by Eq. (383) [7]. First, for convenience we repeat the fundamental definition

$$u(x,t) \;=\; A(x,t)e^{i\varphi(x,t)} \tag{865}$$

$$S(k,t) = \frac{1}{\sqrt{2\pi}}\int u(x,t)\; e^{-ikx}\,dx = B(k,t)e^{i\psi(k,t)} \tag{866}$$

$$F(\omega,x) = \frac{1}{\sqrt{2\pi}}\int u(x,t)\; e^{-j\omega t}\,dt = B(\omega,x)e^{i\psi(\omega,x)} \tag{867}$$

$$G(k,\omega) \;=\; \frac{1}{2\pi}\iint u(x,t)\,e^{-j\omega t - jkx}\,dt\,dx = L(k,\omega)\,e^{i\psi(k,\omega)} \tag{868}$$

and also repeat the four different ways one can write the four dimensional distribution

$$W(x,k,t,\omega) = \left(\frac{1}{2\pi}\right)^2 \iint u^*(x-\frac{1}{2}\tau_x, t-\frac{1}{2}\tau)\, u(x+\frac{1}{2}\tau_x, t+\frac{1}{2}\tau)\, e^{-i\tau_x k - i\tau\omega}\, d\tau\, d\tau_x \tag{869}$$

$$= \left(\frac{1}{2\pi}\right)^2 \iint G^*(k+\frac{1}{2}\theta_x, \omega+\frac{1}{2}\theta)\, G(k-\frac{1}{2}\theta_x, \omega-\frac{1}{2}\theta)\, e^{-j\theta_x x - j\theta t}\, d\theta\, d\theta_x \tag{870}$$

$$= \left(\frac{1}{2\pi}\right)^2 \iint S^*(k+\frac{1}{2}\theta_x, t-\frac{1}{2}\tau)\, S(k-\frac{1}{2}\theta_x, t-\frac{1}{2}\tau)\, e^{-j\theta_x x - j\tau\omega}\, d\theta\, d\tau \tag{871}$$

$$= \left(\frac{1}{2\pi}\right)^2 \iint F^*(x-\frac{1}{2}\tau_x, \omega+\frac{1}{2}\theta) F(x+\frac{1}{2}\tau_x, \omega-\frac{1}{2}\theta)\, e^{-jt\theta - j\tau_x k}\, d\theta\, d\tau_x \tag{872}$$

There are 4 three dimensional marginals

$$P(x,k,t) = \int C(x,k,t,\omega)\, d\omega \tag{873}$$

$$= \frac{1}{2\pi} \int u^*(x-\frac{1}{2}\tau_x, t)\, u(x+\frac{1}{2}\tau_x, t)\, e^{-j\tau_x k}\, d\tau_x \tag{874}$$

$$= \frac{1}{2\pi} \int S^*(k+\frac{1}{2}\theta_x, t)\, S(k-\frac{1}{2}\theta_x, t)\, e^{-j\theta_x x}\, d\theta_x \tag{875}$$

$$P(x,k,\omega) = \int C(x,k,t,\omega)\, dt \tag{876}$$

$$= \frac{1}{2\pi} \int G^*(k+\frac{1}{2}\theta_x, \omega)\, G(k-\frac{1}{2}\theta_x, \omega)\, e^{-j\theta_x x}\, d\theta_x \tag{877}$$

$$= \frac{1}{2\pi} \int F^*(x-\frac{1}{2}\tau_x, \omega) F(x+\frac{1}{2}\tau_x, \omega)\, e^{-j\tau_x k}\, d\tau_x \tag{878}$$

$$P(x,t,\omega) = \int C(x,k,t,\omega)\, dk \tag{879}$$

$$= \frac{1}{2\pi} \int u^*(x, t-\frac{1}{2}\tau)\, u(x, t+\frac{1}{2}\tau)\, e^{-j\tau\omega}\, d\tau \tag{880}$$

$$= \frac{1}{2\pi} \int F^*(x, \omega+\frac{1}{2}\theta) F(x, \omega-\frac{1}{2}\theta)\, e^{-jt\theta}\, d\theta \tag{881}$$

$$P(k,t,\omega) = \int C(x,k,t,\omega)\, dx \tag{882}$$

$$= \frac{1}{2\pi} \int G^*(k, \omega+\frac{1}{2}\theta)\, G(k, \omega-\frac{1}{2}\theta)\, e^{-j\theta t}\, d\theta \tag{883}$$

$$= \frac{1}{2\pi} \int S^*(k, t-\frac{1}{2}\tau)\, S(k, t-\frac{1}{2}\tau)\, e^{-j\tau\omega}\, d\theta \tag{884}$$

There are six two dimensional marginals given MUST BE CHECKED

$$P(t,\omega) = \int W(x,k,t,\omega)\, dx\, dk \tag{885}$$

$$= \frac{1}{2\pi} \iint u^*(x, t - \frac{1}{2}\tau)\, u(x, t + \frac{1}{2}\tau)\, e^{-i\tau\omega}\, d\tau\, dx \tag{886}$$

$$P(x,k) = \int W(x,k,t,\omega)\, dt\, d\omega \tag{887}$$

$$= \frac{1}{2\pi} \iint u^*(x - \frac{1}{2}\tau_x, t)\, u(x + \frac{1}{2}\tau_x, t)\, e^{-i\tau_x k}\; d\tau_x dt \tag{888}$$

$$P(k,t) = \int W(x,k,t,\omega)\, d\omega\, dk = |S(k,t)\,|^2 \tag{889}$$

$$P(k,\omega) = \int W(x,k,t,\omega)\, dt\, dx = \; |G(k,\omega)\,|^2 \tag{890}$$

$$P(t,x) = \int W(x,k,t,\omega)\, d\omega\, dk = |\, u(x,t)\,|^2 \tag{891}$$

$$P(x,\omega) = \int W(x,k,t,\omega)\, dt\, dk = \; |F(\omega,x)\,|^2 \tag{892}$$

There are 4 one dimensional marginals

$$P(t) = \int W(x,k,t,\omega)\, dx\, dk d\omega = \int |\, u(x,t)\,|^2 dx \tag{893}$$

$$P(x) = \int W(x,k,t,\omega)\, dt\, d\omega dk = \int |\, u(x,t)\,|^2 dt \tag{894}$$

$$P(k) = \int W(x,k,t,\omega)\, d\omega\, dk dt = \int |S(k,t)\,|^2\; dt \tag{895}$$

$$P(\omega) = \int W(x,k,t,\omega)\, dt\, dk dt = \int \; |F(\omega,x)\,|^2\; dx \tag{896}$$

Frequency moments. The first conditional moment of frequency is

$$\langle\omega\rangle_{x,k,t} = \int \omega C(x,k,t,\omega)\,d\omega \tag{897}$$

$$= \left(\frac{1}{2\pi}\right)^2 \iiint \omega u^*(x-\frac{1}{2}\tau_x, t-\frac{1}{2}\tau)\,u(x+\frac{1}{2}\tau_x, t+\frac{1}{2}\tau)\,e^{-i\tau_x k - i\tau\omega}\,d\tau\,d\tau_x\,d\omega \tag{898}$$

$$= -\frac{1}{2\pi i}\int \frac{\partial}{\partial\tau}u^*(x-\frac{1}{2}\tau_x, t-\frac{1}{2}\tau)u(x+\frac{1}{2}\tau_x, t+\frac{1}{2}\tau)\,e^{-i\tau_x k}|_{\tau=0}\,d\tau_x \tag{899}$$

$$= -\frac{1}{2\pi}\frac{1}{2i}\int \left[\frac{\partial}{\partial t}u^*(x-\frac{1}{2}\tau_x, t)\,u(x+\frac{1}{2}\tau_x, t)\ - u^*(x-\frac{1}{2}\tau_x, t)\,\frac{\partial}{\partial t}u(x+\frac{1}{2}\tau_x, t)\right]e^{-i\tau_x k}\,d\tau_x \tag{900}$$

$$= -\frac{1}{2\pi}\frac{1}{2i}\int \Big[A(x+\frac{1}{2}\tau_x, t)\frac{\partial}{\partial t}A(x-\frac{1}{2}\tau_x, t) - A(x-\frac{1}{2}\tau_x, t)\frac{\partial}{\partial t}A(x+\frac{1}{2}\tau_x, t)$$
$$-iA(x+\frac{1}{2}\tau_x, t)A(x-\frac{1}{2}\tau_x, t)\frac{\partial}{\partial t}\left\{\varphi(x+\frac{1}{2}\tau_x, t)+\varphi(x-\frac{1}{2}\tau_x, t)\right\}\Big]$$
$$e^{i\phi(x+\frac{1}{2}\tau_x,t)-i\phi(x-\frac{1}{2}\tau_x,t)-i\tau_x k}\,d\tau_x \tag{901}$$

Also,

$$\langle\omega\rangle_{x,t} = \int \omega C(x,k,t,\omega)\,d\omega dk \tag{902}$$

Using Eq. (897) we have that

$$\langle\omega\rangle_{x,t} = \int \omega C(x,k,t,\omega)\,d\omega dk \tag{903}$$

$$= \frac{1}{2\pi i}\int \frac{\partial}{\partial\tau}u^*(x-\frac{1}{2}\tau_x, t-\frac{1}{2}\tau)u(x+\frac{1}{2}\tau_x, t+\frac{1}{2}\tau)\,e^{-i\tau_x k}|_{\tau=0}\,d\tau_x dk \tag{904}$$

$$= \frac{1}{i}\int \delta(\tau_x)\frac{\partial}{\partial\tau}u^*(x-\frac{1}{2}\tau_x, t-\frac{1}{2}\tau)u(x+\frac{1}{2}\tau_x, t+\frac{1}{2}\tau)|_{\tau=0}\,d\tau_x \tag{905}$$

$$= \frac{1}{i}\int \frac{\partial}{\partial\tau}u^*(x, t-\frac{1}{2}\tau)u(x, t+\frac{1}{2}\tau)|_{\tau=0}\,d\tau_x \tag{906}$$

which evaluates to

$$\langle\omega\rangle_{x,t} = A^2(x,t)\frac{\partial}{\partial t}\varphi(x,t) \tag{907}$$

Conditional Position. The average position is similarly given

$$\langle\, x \,\rangle_{k,t,\omega} = \int x\, C(x,k,t,\omega)\, dx \tag{908}$$

$$= \frac{1}{2\pi}\frac{1}{2i}\int \left(S^*(k,t-\frac{1}{2}\tau)\, \frac{\partial S(k,t+\frac{1}{2}\tau)}{\partial k} - \frac{\partial S^*(k,t-\frac{1}{2}\tau)}{\partial k} S(k,t+\frac{1}{2}\tau) \right) e^{-j\tau\omega}\, d\tau \tag{909}$$

$$= \frac{1}{2\pi}\frac{1}{2i}\int B(k,t-\frac{1}{2}\tau)\frac{\partial B(k,t+\frac{1}{2}\tau)}{\partial k} - B(k,t+\frac{1}{2}\tau)\frac{\partial B(k,t-\frac{1}{2}\tau)}{\partial k} \tag{910}$$

$$+\, jB(k,t+\frac{1}{2}\tau)B(k,t-\frac{1}{2}\tau)\frac{\partial}{\partial k}\left\{ \psi(k,t+\frac{1}{2}\tau) + \psi(k,t-\frac{1}{2}\tau) \right\} e^{i\psi(k,t+\frac{1}{2}\tau)-j\psi(k,t-\frac{1}{2}\tau)-j\tau\omega}\, d\tau \tag{911}$$

To obtain the average position for given wave number and time we have

$$\langle\, x \,\rangle_{k,t} = \int x\, C(x,k,t,\omega)\, dx\, d\omega \tag{912}$$

$$= \frac{1}{2i}\int \left(S^*(k,t)\, \frac{\partial S(k,t)}{\partial k} - \frac{\partial S^*(k,t)}{\partial k}\, S(k,t) \right) e^{-j\tau\omega}\, d\tau \tag{913}$$

$$= B^2(k,t)\, \frac{\partial}{\partial k}\psi(k,t) \tag{914}$$

We now further average to obtain

$$\langle\, x \,\rangle_t = \int B^2(k,t)\, \frac{\partial}{\partial k}\psi(k,t)\, dk = \langle\, \frac{\partial}{\partial k}\psi(k,t) \,\rangle$$

Covariance. We now calculate the covariance. Consider $\langle k\omega \rangle_{x,t}$

$$\langle k\omega \rangle_{x,t} = \int k\omega C(x,k,t,\omega)\, dk\, d\omega \tag{915}$$

$$= \int k \langle \omega \rangle_{x,k,t}\, dk \tag{916}$$

$$= A^2(x,t)\frac{\partial \varphi(x,t)}{\partial x}\frac{\partial \varphi(x,t)}{\partial t} \tag{917}$$

The local covariance of frequency and spatial frequency is therefore

$$\mathrm{Cov}_{x,t}(k\omega) = A^2(x,t)\left(\frac{\partial \varphi(x,t)}{\partial t}\frac{\partial \varphi(x,t)}{\partial x} - \frac{\partial \varphi(x,t)}{\partial t} - \frac{\partial \varphi(x,t)}{\partial x} \right) \tag{918}$$

and the global covariance is hence

$$\mathrm{Cov}(k\omega) = \iint A^2(x,t)\left(\frac{\partial \varphi(x,t)}{\partial t}\frac{\partial \varphi(x,t)}{\partial x} - \frac{\partial \varphi(x,t)}{\partial t} - \frac{\partial \varphi(x,t)}{\partial x} \right) dx\, dt \tag{919}$$

As discussed previously, we can think of $\frac{\partial \varphi(x,t)}{\partial t}$ and $\frac{\partial \varphi(x,t)}{\partial x}$ as the frequency and spatial frequency in the x,t representation. Using this idea for example we can immediately write

$$\langle k\rangle_{x,t} = A^2(x,t)\frac{\partial \varphi(x,t)}{\partial x} \tag{920}$$

In the spectral domain we have

$$\langle xt\rangle_{k,\omega} = B^2(k,\omega)\frac{\partial \psi(k,\omega)}{\partial k}\frac{\partial \psi(k,\omega)}{\partial \omega} \tag{921}$$

and again one can think of x,t as $\frac{\partial \psi(k,\omega)}{\partial k}\frac{\partial \psi(k,\omega)}{\partial \omega}$.

E Appendix: Asymptotics for pulse propagation

In the text we have given formulas for the instantaneous frequency where we have based our approach on the classical asymptotic approximation. We repeat here the classical asymptotic result. The pulse is given by (exactly)

$$u(x,t) \ = \ \frac{1}{\sqrt{2\pi}}\int S(k,0)\ e^{ikx-iW(k)t}\,dk \tag{922}$$

where $S(k,0)$ is the initial spatial spectrum and it is calculated from the initial pulse. The asymptotic solution as standardly given is

$$u_a(x,t) \sim S(k_s,0)\sqrt{\frac{1}{tW''(k_s)}}e^{ik_sx-iW(k_s)t-i\pi\mathrm{sgn}\,W''/4} \tag{923}$$

where k_s is obtained from

$$W'(k_s) = \frac{x}{t} \tag{924}$$

In deriving the formulas for instantaneous frequency we wrote

$$u_a(x,t) \sim B(k,0)e^{i\psi(k,0)}\sqrt{\frac{1}{tW''(k_s)}}e^{ik_sx-iW(k_s)t-i\pi\mathrm{sgn}\,W''/4} \tag{925}$$

where $B(k,0)$ and $\psi(k,0)$ are the initial amplitude and phase of the spectrum. The asymptotic amplitude and phase of the pulse are therefore

$$A_a(x,t) = |S(k_s,0)|\sqrt{\frac{1}{tW''(k_s)}} \tag{926}$$

$$\varphi_a(x,t) = \psi(k_s,0) + k_sx - W(k_s)t - \pi\mathrm{sgn}\,W''/4 \tag{927}$$

and we obtained our formulas for instantaneous frequency by differentiating asymptotic phase appropriately. We repeat those results here

$$\omega_i(x,t) = -\frac{\partial}{\partial t}\varphi(x,t) = \frac{x}{t^2 W''(k_s)}\frac{d\psi}{dk_s} + W(k_s) \tag{928}$$

$$k_i(x,t) = \frac{\partial}{\partial x}\varphi(x,t) = \frac{1}{tW''(k_s)}\frac{d\psi(k,0)}{dk_s} + k_s \tag{929}$$

We now show that a different approach is more appropriate and gives considerable better answer. This has recently been done with P. Loughlin [14]. Our approach consists of two steps. First we derive a more suitable asymptotic approximation then the classical one. Secondly, using this new asymptotic approximation we calculate the Instantaneous frequency.

To derive the new aproximation we take the initial spectral phase into account in deriving the asymptotic formula. We write

$$u(x,t) \;=\; \frac{1}{\sqrt{2\pi}}\int B(k,0)\; e^{ikx-iW(k)t+i\psi(k,0)}\,dk \tag{930}$$

In the stationary phase method one assumes that $e^{ikx-iW(k)t+i\psi(k,0)}$ is rapidly oscillating and the amplitude is relatively slowly varying. Hence, for most regions of integration there will be cancellations. However if there is a region where the phase is not oscillating then that region will contribute the greatest part to the integral. That region is where the derivative of the phase is zero

$$\frac{\partial}{\partial k}[kx - W(k)t + \psi(k,0)] = 0 \tag{931}$$

This gives

$$x - W'(k)t + \psi'(k,0) = 0 \tag{932}$$

or equivalently,

$$W'(k) = \frac{x+\psi'(k,0)}{t} \tag{933}$$

One solves this equation for k and writes the solution as k_s. Notice now that k_s is no longer a function of x/t as would be the case if we did not take into account the phase of the initial spectrum. Since we assume that the *amplitude* is slowly varying we have

$$u_a(x,t) \;\sim\; \frac{1}{\sqrt{2\pi}}B(k_s,0)\int \; e^{ikx-iW(k)t+i\psi(k,0)}\,dk \tag{934}$$

The phase is now expanded in a series about the stationary point k_s and one keeps terms only up to the quadratic one. In particular

$$kx - W(k)t + \psi(k,0) \sim k_s x - W(k_s)t + \psi(k_s,0) + [x - W'(k_s)t + \psi'(k_s,0)](k-k_s) \quad (935)$$

$$+\frac{1}{2}[-W''(k_s)t + \psi''(k_s,0)](k-k_s)^2 \quad (936)$$

$$= k_s x - W(k_s)t + \psi(k_s,0) + \frac{1}{2}[-W''(k_s)t + \psi''(k_s,0)](k-k_s)^2 \quad (937)$$

and therefore

$$u_a(x,t) \sim \frac{1}{\sqrt{2\pi}} B(k_s,0) \int \exp i[k_s x - W(k_s)t + \psi(k_s,0) + \frac{1}{2}[-W''(k_s)t + \psi''(k_s,0)](k-k_s)^2 \,]\, dk \quad (938)$$

$$= \frac{1}{\sqrt{2\pi}} S(k_s,0) e^{i[k_s x - W(k_s)t]} \int \exp i[\frac{1}{2}\{-W''(k_s)t + \psi''(k_s,0)\}(k-k_s)^2 \,]\, dk \quad (939)$$

Using,

$$\int e^{jat^2/2} dt = \sqrt{\frac{2\pi}{-ja}} = \sqrt{\frac{2\pi}{|a|}} e^{j\pi\mu_a/4} \quad (940)$$

where μ_a is the sign of a

$$\mu_a = \mathrm{sgn}(a) \quad (941)$$

we have

$$u_a(x,t) \sim \frac{1}{\sqrt{2\pi}} S(k_s,0) e^{i[k_s x - W(k_s)t]} \int \exp i[\frac{1}{2}\{-W''(k_s)t + \psi''(k_s,0)\}(k-k_s)^2 \,]\, dk \quad (942)$$

$$\sim S(k_s,0) \sqrt{\frac{1}{tW''(k_s) - \psi''(k_s,0)}} e^{ik_s x - iW(k_s)t - i\pi \mathrm{sgn}\,[W''t - \psi''(k_s,0)]/4} \quad (943)$$

Note the following. From a functional point of view there appears to be only a difference in the amplitude and not the phase when one compares Eq.(943 with Eq. (925). However it must be emphasized that there is a difference as to the value of the stationary points. Namely, for this case we must solve Eq. (933) rather than Eq. (924) for k_s.

Instantaneous frequency. We now obtain the instantaneous frequency. The phase is

$$\varphi_a(x,t) = \psi(k_s,0) + k_s x - W(k_s)t - \pi \mathrm{sgn}\,[W''t - \psi''(k_s,0)]/4 \quad (944)$$

Differentiating the phase, $\varphi_a(x,t)$, we have

$$\omega_i(x,t) = -\frac{\partial}{\partial t}\varphi_a(x,t) \tag{945}$$

$$= -\left[\frac{d\psi}{dk_s}\frac{\partial k_s}{\partial t} + x\frac{\partial k_s}{\partial t} - t\frac{dW(k_s)}{dk_s}\frac{\partial k_s}{\partial t}\right] + W(k_s) \tag{946}$$

$$= -\left[\frac{d\psi}{dk_s} + x - t\frac{dW(k_s)}{dk_s}\right]\frac{\partial k_s}{\partial t} + W(k_s) \tag{947}$$

$$= -\left[\frac{d\psi}{dk_s} + x - t\left(\frac{x+\psi'(k,0)}{t}\right)\right]\frac{\partial k_s}{\partial t} + W(k_s) \tag{948}$$

But since

$$W'(k) = \frac{x+\psi'(k,0)}{t} \tag{949}$$

the first term is identically zero and hence

$$\omega_i(x,t) = W(k_s) \tag{950}$$

Also,

$$k_i(x,t) = \frac{\partial}{\partial x}\varphi_a(x,t) \tag{951}$$

$$= \frac{d\psi}{dk_s}\frac{\partial k_s}{\partial x} + x\frac{\partial k_s}{\partial x} + k_s - t\frac{dW(k_s)}{dk_s}\frac{\partial k_s}{\partial x} \tag{952}$$

$$= \left[\frac{d\psi}{dk_s} + x - t\frac{dW(k_s)}{dk_s}\right]\frac{\partial k_s}{\partial x} + k_s \tag{953}$$

$$= \left[\frac{d\psi}{dk_s} + x - t\left(\frac{x+\psi'(k,0)}{t}\right)\right]\frac{\partial k_s}{\partial x} + k_s \tag{954}$$

and therefore

$$k_i(x,t) = k_s \tag{955}$$

Example. We now take an example to illustrate the differences. We shall do it three different ways, exact, using the classical asymptotic approach, and the modified asymptotic approach discussed above. At $t=0$ we take

$$u(x,0) = e^{i\beta x^2/2} \tag{956}$$

and the initial spectrum is

$$S(k,0) = \sqrt{\frac{i}{\beta}}\exp\left[-i\frac{k^2}{2\beta}\right] \tag{957}$$

For the dispersion relation we take

$$W(k) = \gamma k^2/2 \tag{958}$$

Exact. The time dependent spectrum is

$$S(k,t) = \sqrt{\frac{i}{\beta}} \exp\left[-i\frac{k^2}{2\beta} - iW(k)t\right] \tag{959}$$

giving

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int S(k,t) e^{ikx} && (960) \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{-i\beta}} \int \exp\left[-i\frac{k^2}{2\beta} - \frac{i\gamma k^2 t}{2} + ikx\right] dk && (961) \\ &= \frac{1}{\sqrt{1+\gamma\beta t}} \exp\left[i\frac{\beta}{2}\frac{x^2}{1+\gamma\beta t}\right] && (962) \end{aligned}$$

The exact phase is therefore

$$\varphi(x,t) = \frac{\beta}{2}\frac{x^2}{1+\gamma\beta t} \tag{963}$$

and the exact instantaneous frequency is

$$\omega_i(x,t) = -\frac{\partial}{\partial t}\varphi(x,t) = \gamma\frac{\beta^2 x^2}{2(1+\gamma\beta t)^2} \tag{964}$$

and the exact spatial instantaneous frequency is

$$k_i(x,t) = \frac{\partial}{\partial t}\varphi(x,t) = \frac{\beta x}{1+\gamma\beta t} \tag{965}$$

Standard asymptotic method. The stationary points are obtained from

$$W'(k) = \gamma k = \frac{x}{t} \tag{966}$$

and hence,

$$k_s = \frac{x}{\gamma t} \tag{967}$$

giving

$$\begin{aligned} W(k_s) &= \frac{x^2}{2\gamma t^2} && (968) \\ k_s x - W(k_s)t &= \frac{x^2}{2\gamma t} && (969) \end{aligned}$$

Therefore,

$$\begin{aligned} u_a(x,t) &\sim S(k_s,0)\sqrt{\frac{1}{tW''(k_s)}} e^{ik_s x - iW(k_s)t - i\pi \operatorname{sgn} K''/4} && (970) \\ &= \sqrt{\frac{1}{\gamma\beta t}} \exp\left[-i\frac{x^2}{2\beta\gamma^2 t^2} + i\frac{x^2}{2\gamma t}\right] && (971) \end{aligned}$$

The phase is

$$\varphi_a(x,t) = -\frac{x^2}{2\beta\gamma^2 t^2} + \frac{x^2}{2\gamma t}$$

and differentiating we obtain

$$\omega_i(x,t) = -\frac{x^2}{\beta\gamma^2 t^3} + \frac{x^2}{2\gamma t^2} \tag{972}$$

$$k_i(x,t) = -\frac{x}{\beta\gamma^2 t^2} + \frac{x}{\gamma t} \tag{973}$$

As a check we also calculate the instantaneous frequencies by

$$\omega_i(x,t) = \frac{x}{t^2 W''(k_s)} \frac{d\psi}{dk_s} + W(k_s) \tag{974}$$

But

$$\frac{d\psi}{dk_s} = -\frac{k}{\beta} = -\frac{x}{\beta\gamma t} \tag{975}$$

and therefore

$$\omega_i(x,t) = -\frac{x}{\gamma t^2}\frac{x}{\beta\gamma t} + \frac{x^2}{2\gamma t^2} \tag{976}$$

$$= -\frac{x^2}{\beta\gamma^2 t^3} + \frac{x^2}{2\gamma t^2} \tag{977}$$

Also,

$$k_i(x,t) = \frac{1}{tW''(k_s)}\frac{d\psi}{dk_s} + k_s \tag{978}$$

$$= -\frac{x}{\beta\gamma^2 t^2} + \frac{x}{\gamma t} \tag{979}$$

As can be seen these are approximations to the exact answers.

Modified Asymptotic method. We have to solve

$$W'(k) = k\gamma = \frac{x + \psi'(k,0)}{t} \tag{980}$$

That is,

$$k_s\gamma = \frac{x - k_s/\beta}{t} \tag{981}$$

and therefore

$$k_s = \frac{\beta x}{1 + \gamma\beta t} \tag{982}$$

(Notice that this happens to be the exact instantaneous spatial frequency, but we do not take that into account.)

First, we calculate $W(k_s)$ and $k_s x - W(k_s)t$. In particular

$$W(k_s) = \gamma k_s^2/2 \tag{983}$$

$$= \frac{\gamma}{2}\left(\frac{\beta x}{1+\gamma\beta t}\right)^2 \tag{984}$$

$$= \frac{\gamma\beta^2 x^2}{2(1+\gamma\beta t)^2} \tag{985}$$

and also

$$k_s x - W(k_s)t = \frac{\beta x}{1+\gamma\beta t}x - \frac{\gamma\beta^2 x^2}{2(1+\gamma\beta t)^2}t \tag{986}$$

$$= \frac{\beta x^2}{1+\gamma\beta t}\left(1 - \frac{\gamma\beta t}{2(1+\gamma\beta t)}\right) \tag{987}$$

$$= \frac{\beta x^2}{(1+\gamma\beta t)^2}(1+\gamma\beta t/2) \tag{988}$$

Therefore

$$u_a(x,t) \sim S(k_s,0)\sqrt{\frac{1}{tW''(k_s)-\psi''(k_s,0)}}e^{ik_s x - iW(k_s)t - i\pi \mathrm{sgn}\,[W''t-\psi''(k_s,0)]/4} = \tag{989}$$

$$= \sqrt{\frac{i}{\beta}}\exp\left[-i\frac{k_s^2}{2\beta}\right]\sqrt{\frac{1}{tW''(k_s)-\psi''(k_s,0)}}e^{ik_s x - iW(k_s)t - i\pi \mathrm{sgn}\,[W''t-\psi''(k_s,0)]/4} \tag{990}$$

$$= \sqrt{\frac{1}{\beta}}\sqrt{\frac{1}{t\gamma+1/\beta}}\exp\left[i\frac{\beta x^2}{(1+\gamma\beta t)^2}(1+\gamma\beta t/2) - i\frac{1}{2\beta}\left(\frac{\beta x}{1+\gamma\beta t}\right)^2\right] \tag{991}$$

$$= \sqrt{\frac{1}{\beta}}\sqrt{\frac{1}{t\gamma+1/\beta}}\exp\left[i\frac{\beta x^2}{(1+\gamma\beta t)^2}(1+\gamma\beta t/2) - i\frac{1}{2\beta}\left(\frac{\beta x}{1+\gamma\beta t}\right)^2\right] \tag{992}$$

$$= \sqrt{\frac{1}{\beta}}\sqrt{\frac{1}{t\gamma+1/\beta}}\exp\left[i\frac{\beta x^2}{(1+\gamma\beta t)^2}(1+\gamma\beta t/2-1/2)\right] \tag{993}$$

$$= \sqrt{\frac{1}{1+t\gamma\beta}}\exp\left[i\frac{\beta x^2}{2(1+\gamma\beta t)}\right] \tag{994}$$

But this is the exact answer and hence will give the exact instantaneous frequencies. To verify we use

$$\omega_i(x,t) = W(k_s) = \frac{\gamma\beta^2 x^2}{2(1+\gamma\beta t)^2} \tag{995}$$

and

$$k_i(x,t) = k_s = \frac{\beta x}{1+\gamma\beta t} \tag{996}$$

and indeed these are exact. We therefore see that the approximation we have presented here is more accurate than the standard asymptotic one.

Case B

We write the corresponding equations for case B. The general solution is

$$u(x,t) \;=\; \frac{1}{\sqrt{2\pi}}\int F(\omega,x)\,e^{i\omega t}\,d\omega \tag{997}$$

$$F(\omega,x) \;=\; \frac{1}{\sqrt{2\pi}}\int u(x,t)\,e^{-i\omega t}\,dt \tag{998}$$

with

$$F(\omega,x) = F(\omega,0)\,e^{-iK(\omega)x} \tag{999}$$

and where $F(\omega,0)$ is the spectrum evaluated at $x=0$,

$$F(\omega,0) = \frac{1}{\sqrt{2\pi}}\int u(0,t)\ e^{-i\omega t}\,dt \tag{1000}$$

The standard asymptotic solution is

$$u_a(x,t) \sim F(\omega,0)\sqrt{\frac{1}{xK''(\omega)}}e^{i\omega t-iK(\omega)x-i\pi\mathrm{sgn}\,K''/4}|_{\omega=\omega_s} \tag{1001}$$

where one obtains ω_s from

$$K'(\omega_s) = t/x \tag{1002}$$

The amplitude and phase are

$$|u_a(x,t)| = |F(\omega,0)|\sqrt{\frac{1}{xK''(\omega)}} \tag{1003}$$

$$\varphi_a(x,t) = \psi(\omega,0) - K(\omega)x + \omega t - \pi\mathrm{sgn}\,K''/4 \tag{1004}$$

and the instantaneous frequencies are

$$\omega_i(x,t) = \frac{1}{xK''(\omega)}\frac{\partial\psi}{\partial\omega} + \omega_s \tag{1005}$$

$$k_i(x,t) = -\frac{t}{x^2K''(\omega_s)}\frac{d\psi}{d\omega_s} + K(\omega_s) \tag{1006}$$

We no obtain the new formulas. Using the same approach as before we now have write

$$u(x,t) \;=\; \frac{1}{\sqrt{2\pi}}\int B(\omega,0)\ e^{i\omega t-iK(\omega)x+i\psi(\omega,0)}\,d\omega \tag{1007}$$

and the region where the derivative of the phase is zero is obtained from

$$\frac{\partial}{\partial\omega}[\omega t - K(\omega)x + \psi(\omega,0)] = 0 \tag{1008}$$

which gives

$$t - K'(\omega)x + \psi'(\omega, 0) = 0 \tag{1009}$$

or

$$K'(\omega) = \frac{t + \psi'(\omega, 0)}{x} \tag{1010}$$

Taking the amplitude out of the integral we have

$$u_a(t,x) \sim \frac{1}{\sqrt{2\pi}} B(\omega_s, 0) \int e^{i\omega t - iK(\omega)x + i\psi(\omega,0)}\, d\omega \tag{1011}$$

Expanding the phase about the stationary point ω_s and keeping terms only up to the quadratic one we have

$$\omega t - K(\omega)x + \psi(\omega,0) \sim \omega_s t - K(\omega_s)x + \psi(\omega_s, 0) + \frac{1}{2}[-K''(\omega_s)x + \psi''(\omega_s, 0)](\omega - \omega_s)^2 \tag{1012}$$

and therefore

$$u_a(t,x) \sim \frac{1}{\sqrt{2\pi}} B(\omega_s, 0) \int \exp i[\omega_s t - K(\omega_s)x + \psi(\omega_s, 0) + \frac{1}{2}[-K''(\omega_s)x + \psi''(\omega_s, 0)](\omega - \omega_s)^2\,]\, d\omega \tag{1013}$$

$$= \frac{1}{\sqrt{2\pi}} S(\omega_s, 0) e^{i[\omega_s t - K(\omega_s)x]} \int \exp i[\frac{1}{2}\{-K''(\omega_s)x + \psi''(\omega_s, 0)\}(\omega - \omega_s)^2\,]\, d\omega = \tag{1014}$$

or,

$$u_a(t,x) \sim S(\omega_s, 0) \sqrt{\frac{1}{xK''(\omega_s) - \psi''(\omega_s, 0)}} e^{i\omega_s t - iK(\omega_s)x - i\pi \mathrm{sgn}\,[K''x - \psi''(\omega_s,0)]/4} \tag{1015}$$

The phase is

$$\varphi_a(t,x) = \psi(\omega_s, 0) + \omega_s t - K(\omega_s)x - \pi \mathrm{sgn}\,[K''x - \psi''(\omega_s, 0)]/4 \tag{1016}$$

Differentiating the phase, $\varphi_a(t\ , x\ )$, we have

$$k_i(t,x) = -\frac{\partial}{\partial x}\varphi_a(t,x) \tag{1017}$$

$$= -\left[\frac{d\psi}{d\omega_s}\frac{\partial \omega_s}{\partial x} + t\frac{\partial \omega_s}{\partial x} - x\frac{dK(\omega_s)}{d\omega_s}\frac{\partial \omega_s}{\partial x}\right] + K(\omega_s) \tag{1018}$$

$$= -\left[\frac{d\psi}{d\omega_s} + t - x\frac{dK(\omega_s)}{d\omega_s}\right]\frac{\partial \omega_s}{\partial x} + K(\omega_s) \tag{1019}$$

$$= -\left[\frac{d\psi}{d\omega_s} + t - x\left(\frac{t + \psi'(\omega, 0)}{x}\right)\right]\frac{\partial \omega_s}{\partial x} + K(\omega_s) \tag{1020}$$

However

$$K'(\omega) = \frac{t + \psi'(\omega, 0)}{x} \tag{1021}$$

and hence

$$k_i(t, x) = K(\omega_s) \tag{1022}$$

Also,

$$\begin{aligned}
\omega_i(t, x) &= \frac{\partial}{\partial t}\varphi_a(t, x) && (1023)\\
&= \frac{d\psi}{d\omega_s}\frac{\partial \omega_s}{\partial t} + t\frac{\partial \omega_s}{\partial t} + \omega_s - x\frac{dK(\omega_s)}{d\omega_s}\frac{\partial \omega_s}{\partial t} && (1024)\\
&= \left[\frac{d\psi}{d\omega_s} + t - x\frac{dK(\omega_s)}{d\omega_s}\right]\frac{\partial \omega_s}{\partial t} + \omega_s && (1025)\\
&= \left[\frac{d\psi}{d\omega_s} + t - x\left(\frac{t + \psi'(\omega, 0)}{x}\right)\right]\frac{\partial \omega_s}{\partial t} + \omega_s && (1026)
\end{aligned}$$

which gives

$$\omega_i(t, x) = \omega_s \tag{1027}$$